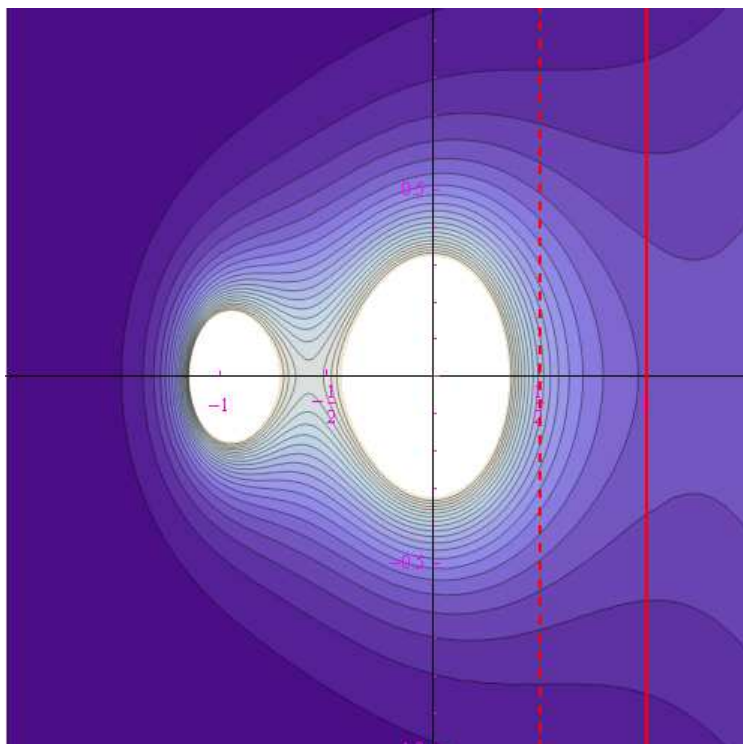


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$$\frac{d}{dx} f(x) = \sum_{k=0}^{+\infty} a_k \int f(x) dx = \oint_{\Gamma} (X dx + Y dy + Z dz)$$

This is a function with the same nontrivial zeros as the Riemann zeta function

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1 Introduction

The Dirichlet series $\sum_{n=1}^{+\infty} n^{-z}$ converges for $\text{Re } z > 1$, and the sum is the *Riemann zeta function*:

$$\zeta(z) = \sum_{n=1}^{+\infty} \frac{1}{n^z}, \quad \text{Re } z > 1 \quad (1)$$

Another notable series that can be expressed through the zeta function is:

$$\sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^z} = (1 - 2^{1-z}) \zeta(z) \quad (2)$$

which converges for $\text{Re } z > 0$.

Since the series are not very “handy” it is preferable to work with integral representations.

2 A remarkable integral representation

In Quantum Statistical Mechanics the following generalized integrals which are not elementary expressible often appear

$$\int_0^{+\infty} \frac{t^{x-1} dt}{e^t \pm 1} \quad (3)$$

having:

$$\begin{aligned} \int_0^{+\infty} \frac{t^{x-1} dt}{e^t + 1} &= (1 - 2^{1-x}) \Gamma(x) \zeta(x), \quad \forall x \in (0, +\infty) \\ \int_0^{+\infty} \frac{t^{x-1} dt}{e^t - 1} &= \Gamma(x) \zeta(x), \quad \forall x \in (1, +\infty) \end{aligned} \quad (4)$$

where $\Gamma(x)$ and $\zeta(x)$ are the Eulerian gamma function and the Riemann zeta function, respectively. Through an elementary change of variable, the first integral becomes

$$\int_{-\infty}^{+\infty} \frac{e^{xt}}{e^{e^t} + 1} dt \quad (5)$$

We define

$$f(x, t) = \frac{e^{xt}}{e^{e^t} + 1}, \quad \begin{cases} x \in (0, 1) & \text{parameter} \\ t \in (-\infty, +\infty) & \text{independent variable} \end{cases} \quad (6)$$

Taking into account the first of (4):

$$\hat{f}(x) \stackrel{\text{def}}{=} \int_{-\infty}^{+\infty} f(x, t) dt = (1 - 2^{1-x}) \Gamma(x) \zeta(x), \quad \forall x \in (0, +\infty), \quad (7)$$

Proceeding by extension to the complex field, we can define the following function:

$$\hat{f}(z) \equiv \hat{f}(x + iy) = \int_{-\infty}^{+\infty} f(x, t) e^{iyt} dt = (1 - 2^{1-z}) \Gamma(z) \zeta(z), \quad \text{Re}(z) > 0 \quad (8)$$

3 The analytic character of $\zeta(z)$, the functional equation and the non-trivial zeros

Riemann found the analytic extension (or *holomorphic extension*) of the sum of the Dirichlet series (1) over all \mathbb{C} except the point $z = 1$, which turns out to be a simple pole with residue 1.

The aforesaid analytical extension is represented by the following functional equation [1]:

$$\pi^{-\frac{z}{2}} \Gamma\left(\frac{z}{2}\right) \zeta(z) = \pi^{\frac{z-1}{2}} \Gamma\left(\frac{1-z}{2}\right) \zeta(1-z) \quad (9)$$

The *non-trivial zeros* of $\zeta(z)$ fall in the *critical strip* [1]-[2] of the complex plane defined by

$$A = \{z \in \mathbb{C} \mid 0 < \operatorname{Re} z < 1, -\infty < \operatorname{Im} z < +\infty\} \quad (10)$$

Proposition 1

$$|(1 - 2^{1-z}) \Gamma(z)| > 0, \quad \forall z \in A \quad (11)$$

$$z_0 \in A \mid \zeta(z_0) = 0 \iff \zeta(1 - z_0) = 0 \quad (12)$$

Proof. The inequality (11) derives from the fact that the gamma function has no zeros [3], while $1 - 2^{1-z}$ is manifestly zero-free in A .

(12) is a consequence of the functional equation (9). ■

From the proposition just proved it follows $f(z)$ and $\zeta(z)$ have the same (non-trivial) zeros. Dalla (12) segue che gli zeri sono simmetrici rispetto alla retta $\operatorname{Re} s = 1/2$. Furthermore, it can be observed that $\zeta(z^*) = \zeta(z)^*$ where $*$ denotes the complex conjugate. This implies that the nontrivial zeros are symmetric about the real axis (see fig. 1).

The line $\operatorname{Re} s = 1/2$ is called the *critical line*. Hardy [1]-[2] proved that infinitely many zeros fall on this line.

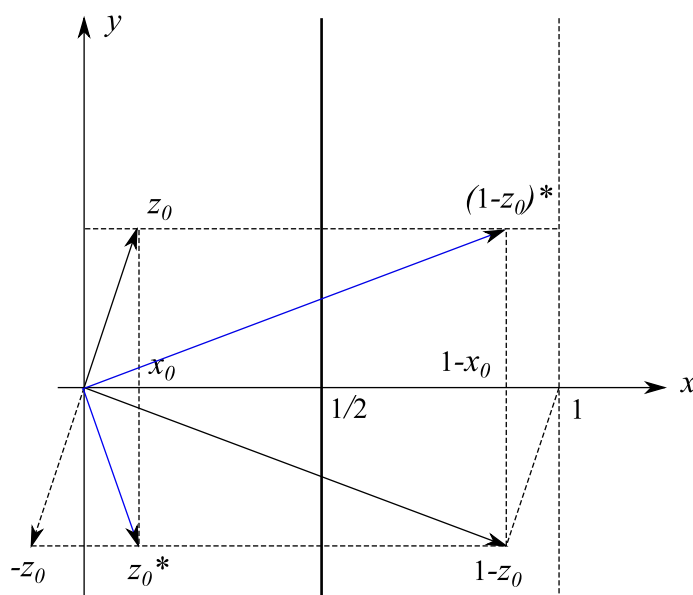


Figure 1: Symmetry of the distribution of non trivial zeros.

4 Riemann Hypothesis

4.1 Fourier Transform

From (8) we see that for a given $x \in (0, 1)$ the complex function $f(x + iy)$ is the Fourier transform of (6).

Conjecture 2 (Riemann Hypothesis)

The non-trivial zeros of the function

$$\hat{f}(x + iy) = \int_{-\infty}^{+\infty} f(x, t) e^{iyt} dt \quad (13)$$

have real part $x = 1/2$.

Let us first study the behavior of the function $f(x, t)$ (given by (6)) which for each value of the parameter $x \in (0, 1)$ is defined in $(-\infty, +\infty)$.

Sign and intersections with the axes

It turns out $g(x, t) > 0, \forall t \in (-\infty, +\infty)$ for which the graph of f lies in the semi-plane of the positive ordinates. It does not intersect the abscissa axis, while it does intersect the ordinate axis at $(0, (e + 1)^{-1})$.

Behavior at extremes

After calculations:

$$\lim_{t \rightarrow +\infty} f(x, t) = 0^+, \quad \forall x \in (0, 1)$$

The order of infinitesimal:

$$\lim_{t \rightarrow +\infty} t^\alpha f(x, t) = 0^+, \quad \forall \alpha > 0 \quad (\text{infinitesimal of infinitely large order}) \quad (14)$$

$$\lim_{t \rightarrow -\infty} f(x, t) = \begin{cases} \frac{1}{2}^-, & \text{if } x = 0 \\ 0^+, & \text{if } x > 0 \end{cases}$$

Precisely:

$$\lim_{t \rightarrow -\infty} t^\alpha f(x > 0, t) = 0^+, \quad \forall \alpha > 0 \quad (15)$$

Conclusion: for $|t| \rightarrow +\infty$ the function $f(x > 0, t)$ is an infinitesimal of order infinitely large, provided that it is $x > 0$.

First derivative

$$f'(x, t) \equiv \frac{\partial}{\partial t} f(x, t) = \frac{e^{xt} [x(e^{et} + 1) - e^{t+et}]}{(e^{et} + 1)^2}$$

For $x = 0$

$$f'(0, t) = -\frac{e^{t+et}}{(e^{et} + 1)^2} < 0, \quad \forall t \in (-\infty, +\infty)$$

so the function is strictly decreasing.

For $x > 0$

$$f'(x, t) = 0 \iff x(e^{et} + 1) - e^{t+et} = 0 \quad (16)$$

which is solved numerically. After calculations, the root of the (16) è

$$0 < x < 1 \implies t_*(x) \in [\sim -6.32, 0.2]$$

Some values for assigned $x \in (0, 1)$:

$$\begin{aligned} t_*\left(\frac{1}{5}\right) &\simeq -1.07 \\ t_*\left(\frac{1}{4}\right) &\simeq -0.88 \\ t_*\left(\frac{1}{2}\right) &\simeq -0.30 \\ t_*\left(\frac{2}{3}\right) &\simeq -0.07 \\ t_*\left(\frac{3}{4}\right) &\simeq 0.02 \end{aligned}$$

The sign is

$$\begin{aligned} -\infty < t < t_*(x) &\implies f'(x, t) > 0 \\ t_*(x) < t < +\infty &\implies f'(x, t) < 0 \end{aligned}$$

Hence the function is strictly increasing in $(-\infty, t_*(x))$ is strictly decreasing in $(t_*(x), +\infty)$. So $t_*(x)$ is a point of relative maximum for

Second derivative

$$f''(x, t) = \frac{e^{xt} \left[e^{2(e^t+t)} - e^{e^t+2t} + x^2 (1 + e^t)^2 - (2x + 1) (e^{t+e^t} + e^{2e^t+t}) \right]}{(1 + e^{e^t})^3} \quad (17)$$

For $x = 0$

$$f''(0, t) = \frac{e^{2(e^t+t)} - e^{e^t+2t} - (e^{t+e^t} + e^{2e^t+t})}{(1 + e^{e^t})^3}$$

which has a zero in $t'_*(x = 0) \simeq 0.43$. The sign is

$$\begin{aligned} -\infty < t < t'_*(x = 0) &\implies f''(0, t) < 0 \\ t'_*(x = 0) < t < +\infty &\implies f''(0, t) > 0 \end{aligned}$$

It follows that the graph of $f(0, t)$ is convex in $(-\infty, t'_*(x = 0))$ and concave in $(t'_*(x = 0), +\infty)$. So $(0.43, 0.18)$ is an inflection point with an oblique tangent. In fig. 2 we report the graph of $f(0, t)$.

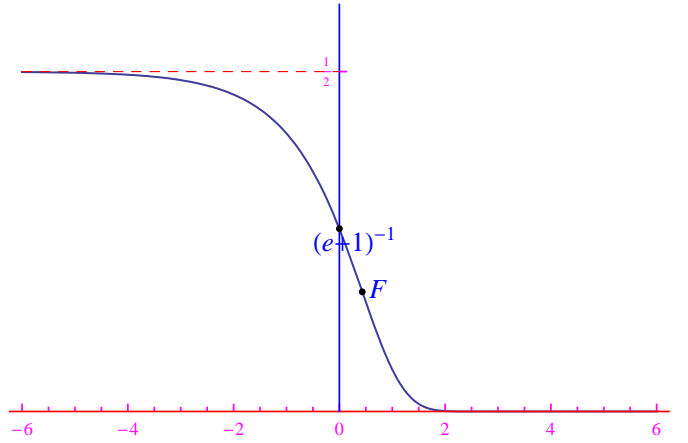
For $x > 0$ we perform a qualitative analysis. The parameter x controls the slope of the graph of $f(t)$ in $(-\infty, 0)$ since

$$\frac{\partial}{\partial t} e^{xt} = x e^{xt}$$

For $t \in (0, +\infty)$ the slope is controlled by e^t in denominator. This implies that the effects of the parameter x are felt for $t \in (-\infty, 0)$, while in $(0, +\infty)$ the trend is practically independent of this parameter. Fig. 3 plots $f(x, t)$ for increasing values of the parameter x starting from $x = 0$.

We rewrite (7)

$$F(x) = \int_{-\infty}^{+\infty} f(x, t) dt \quad (18)$$

Figure 2: Trend of $f(0, t)$.

which *Mathematica* calculates through

$$F(x) = (1 - 2^{1-x}) \Gamma(x) \zeta(x)$$

As previously seen, for $x > 0$ the integrand function is for $t \rightarrow \pm\infty$ an infinitesimal of infinitely large order; so the integral converges. More precisely:

$$F(x) = \int_{-\infty}^0 \frac{e^{xt}}{e^{e^t} + 1} dt + \underbrace{\int_0^{+\infty} \frac{e^{xt}}{e^{e^t} + 1} dt}_{\text{converges } \forall x \in (0,1)}$$

For $x = 0$

$$f(0, t) = \frac{1}{e^{e^t} + 1} \xrightarrow{t \rightarrow -\infty} \frac{1}{2} \implies \int_{-\infty}^0 \frac{dt}{e^{e^t} + 1} = +\infty \implies \lim_{x \rightarrow 0^+} F(x) = +\infty$$

For $x > 0$ the trend in $(-\infty, 0)$ is dominated by e^{xt}

$$\frac{e^{xt}}{e^{e^t} + 1} \xrightarrow{t \rightarrow -\infty} e^{xt}$$

so the integral converges. As x increases in $(-\infty, 0)$ the slope increases, and this favors the convergence of the integral¹, simultaneously decreases the area of the base trapezoid $(-\infty, 0)$ and therefore the value of $F(x)$. This shows that $G(x)$ is strictly decreasing, as confirmed by the graph fig. 4 obtained with *Mathematica*.

¹The parameter x therefore controls the speed of convergence of the integral in the interval $(-\infty, 0)$.

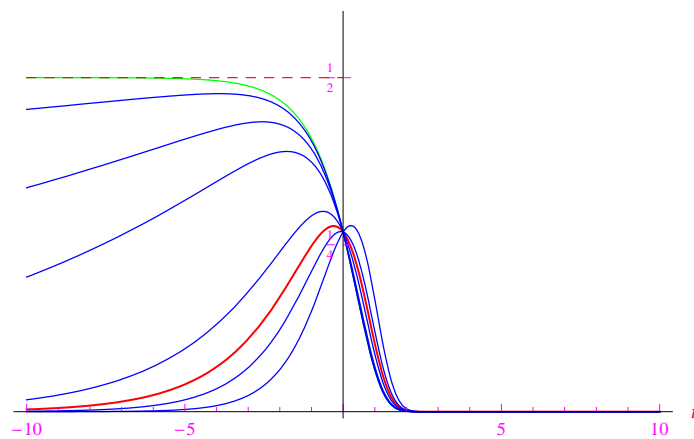


Figure 3: Trend of $f(x, t)$ for different values of x . Curve in green: $x = 0$. The flattest curve towards the ordinate axis is for $x = 1$.

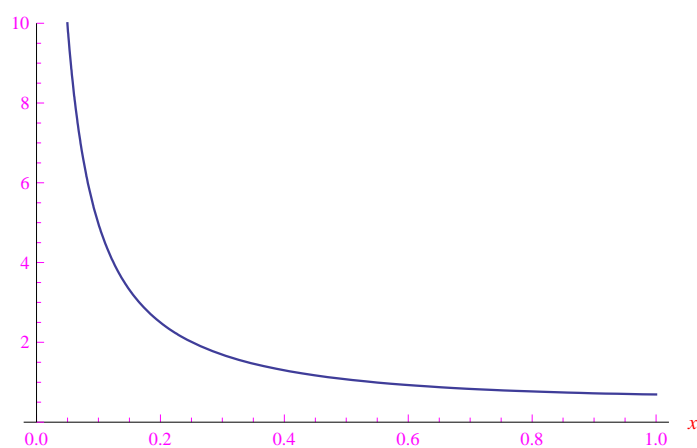


Figure 4: Trend of $F(x)$.

4.2 Zeros of the Fourier Transform

We redefine the variables x, y in α, ω , and then rewrite (13):

$$I(\alpha, \omega) = \int_{-\infty}^{+\infty} \frac{e^{\alpha t}}{e^{e^t} + 1} e^{i\omega t} dt \quad (19)$$

It follows

$$I(\alpha, \omega) = I_-(\alpha, \omega) + I_+(\alpha, \omega) \quad (20)$$

$$I_-(\alpha, \omega) \stackrel{\text{def}}{=} \int_{-\infty}^0 \frac{e^{\alpha t}}{e^{e^t} + 1} e^{i\omega t} dt$$

$$I_+(\alpha, \omega) \stackrel{\text{def}}{=} \int_0^{+\infty} \frac{e^{\alpha t}}{e^{e^t} + 1} e^{i\omega t} dt$$

Even if we are interested in $\alpha \in (0, 1)$, it results:

$$|I_-(\alpha, \omega)| = \int_{-\infty}^0 \frac{e^{\alpha t}}{e^{e^t} + 1} dt = +\infty, \quad \forall \alpha \in (-\infty, 0]; \quad |I_\alpha^{(-)}(\omega)| < +\infty, \quad \forall \alpha \in (0, +\infty)$$

$$|I_+(\alpha, \omega)| < +\infty, \quad \forall \alpha \in (-\infty, +\infty)$$

Conclusion 3 *The variable α conditions the convergence of $I_-(\alpha, \omega)$ but not that of $I_+(\alpha, \omega)$.*

The integral (19) can be seen as:

- complex function of the real variables (α, ω) ;
- complex function of the complex variable $\alpha + i\omega$;
- family of functions of ω , with one real parameter α .

Due to the symmetry property established in the number 3, we can limit the search for zeros in the region:

$$A_1 = \left\{ (\alpha, \omega) \in \mathbb{R}^2 \mid 0 < \alpha < \frac{1}{2}, \quad 0 \leq \omega < +\infty \right\} \quad (21)$$

Ricerca degli zeri:

$$I(\alpha, \omega) = 0 \iff I_-(\alpha, \omega) = -I_+(\alpha, \omega) \implies |I_-(\alpha, \omega)| = |I_+(\alpha, \omega)|$$

Quindi passiamo ai moduli:

$$|I_-(\alpha, \omega)| = \int_{-\infty}^0 \frac{e^{\alpha t}}{e^{e^t} + 1} dt = +\infty, \quad \forall \alpha \in (-\infty, 0]; \quad |I_\alpha^{(-)}(\omega)| < +\infty, \quad \forall \alpha \in (0, +\infty) \quad (22)$$

$$|I_+(\alpha, \omega)| < +\infty, \quad \forall \alpha \in (-\infty, +\infty)$$

$$|I_-(\alpha, \omega)| = \left| \int_{-\infty}^0 \frac{e^{\alpha t}}{e^{e^t} + 1} e^{i\omega t} dt \right| \leq \int_{-\infty}^0 \left| \frac{e^{\alpha t}}{e^{e^t} + 1} e^{i\omega t} \right| dt = \int_{-\infty}^0 \frac{e^{\alpha t}}{e^{e^t} + 1} dt,$$

i.e.

$$|I_-(\alpha, \omega)| \leq \psi_-(\alpha) \quad (23)$$

where

$$\psi_-(\alpha) \stackrel{def}{=} \int_{-\infty}^0 \frac{e^{\alpha t}}{e^{e^t} + 1} dt \quad (24)$$

We have

$$\sup_{\mathbb{R}} \left(\frac{1}{e^{e^t} + 1} \right) = \frac{1}{2} \implies \int_{-\infty}^0 \frac{e^{\alpha t}}{e^{e^t} + 1} dt < \frac{1}{2} \int_{-\infty}^0 e^{\alpha t} dt = \frac{1}{2\alpha}$$

So

$$|I_-(\alpha, \omega)| \leq \psi_-(\alpha) < \frac{1}{2\alpha}, \quad \forall \alpha \in (0, 1), \quad \forall \omega \in \mathbb{R} \quad (25)$$

Dalla prima delle (22):

$$\lim_{\alpha \rightarrow 0^+} |I_-(\alpha, \omega)| = +\infty, \quad \forall \omega \in \mathbb{R} \quad (26)$$

È preferibile lavorare su un piano cartesiano $(O\alpha\beta)$ dove riportiamo le funzioni di α definite più sopra, per assegnati valori della variabile reale ω . Ad esempio, $|I_-(\alpha, \omega)|$ determina una famiglia di curve ad un parametro:

$$\mathcal{F}_{\omega \in \mathbb{R}} = \{\gamma_\omega : \beta = |I_-(\alpha, \omega)|\} \quad (27)$$

aventi in comune l'asintoto verticale $\alpha = 0$.

Notation 4 *Non dobbiamo confondere il piano cartesiano $(O\alpha\beta)$ con il piano complesso definito dalla variabile $\alpha + i\omega$. A titolo di esempio, riportiamo in fig. 5 le curve di livello della funzione $|I_-(\alpha + i\omega,)|$. Si osservi la linea critica e la corrispondente fascia critica.*

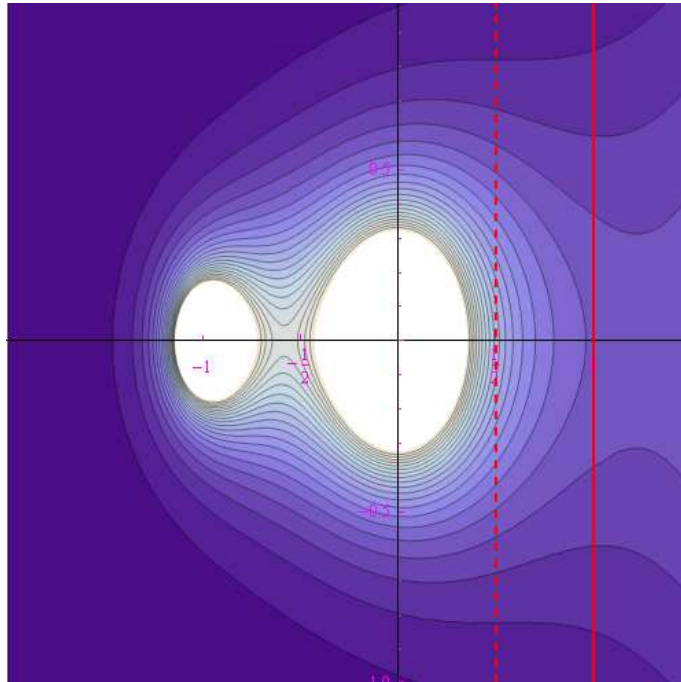


Figure 5: Alcune curve di livello della funzione $|I_-(z = \alpha + i\omega,)|$. Si notino le singolarità in $z = -1$ e in $z = 0$.

La famiglia (27) è limitata superiormente dalla curva

$$\gamma_0 : \beta = |I_-(\alpha, 0)| = \psi_-(\alpha)$$

che è a sua volta limitata da $\beta = \frac{1}{2\alpha}$, come illustrato in fig. 6.

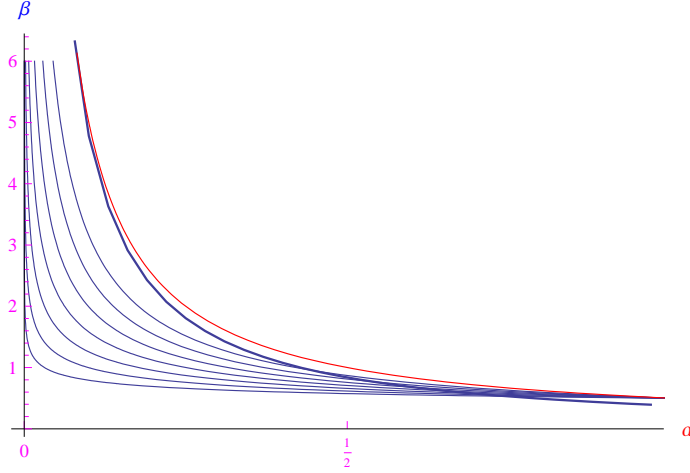


Figure 6: La curva grassettata in blu è $\gamma_0 : \beta = \psi_-(\alpha)$, $\gamma_0 \in \mathcal{F}_\omega$, quella in rosso $\gamma : \beta = \frac{1}{2\alpha}$, $\gamma \notin \mathcal{F}_\omega$.

Dalla (26)

$$\forall M > 0, \exists \Delta_M(\omega) > 0 \mid 0 < \alpha < \Delta_M(\omega) \implies |I_-(\alpha, \omega)| > M \quad (28)$$

Precisamente, per un assegnato $M > 0$, la disuguaglianza

$$\left| \int_{-\infty}^0 \frac{e^{\alpha t}}{e^{e^t} + 1} e^{i\omega t} dt \right| > M$$

è verificata per $0 < \alpha < \Delta_M(\omega)$, cioè in un intorno destro di $\alpha = 0$, la cui ampiezza dipende da ω . Svincoliamoci da ω , considerando

$$\delta_M = \inf_{\mathbb{R}} \{\Delta_M(\omega)\} > 0 \quad (29)$$

per cui

$$\forall M > 0, \exists \delta_M > 0 \mid 0 < \alpha < \delta_M \implies |I_-(\alpha, \omega)| > M, \quad \forall \omega \in \mathbb{R} \quad (30)$$

Ne segue che per un assegnato $M > 0$ la regione $(0, \delta_M) \times [0, M]$ non è attraversata da alcun arco di curva $\gamma_\omega \in \mathcal{F}_\omega$.

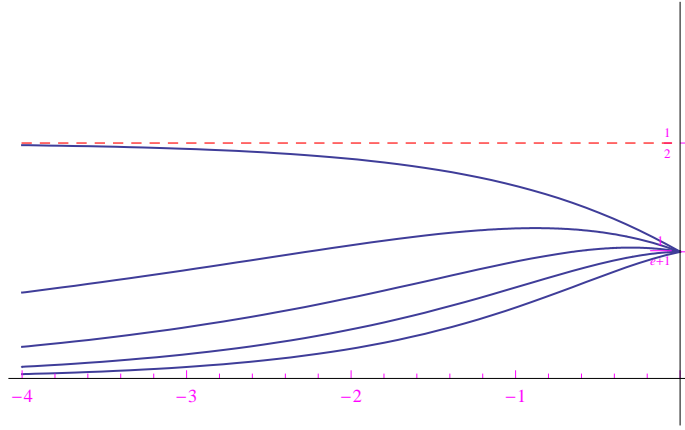
The function $\psi_-(\alpha)$ is monotonically decreasing in $(0, 1)$. This can be deduced from the trend of the integrand function for different values of $\alpha \in (0, 1)$ (fig. 7).

Quindi

$$0 < \alpha < 1 \implies +\infty > \psi_-(\alpha) > \psi_-(1)$$

$\psi_-(\alpha)$ è elementarmente esprimibile solo per $\alpha = 1$:

$$\psi_-(1) = \int_{-\infty}^0 \frac{e^t dt}{e^{e^t} + 1} \stackrel{x = \ln t}{=} \int_0^1 \frac{e^{-x} dx}{1 + e^{-x}} = - \int_0^1 \frac{d}{dx} \ln(1 + e^{-x}) dx$$


 Figure 7: Trend of $\frac{e^{\alpha t}}{e^{e^t} + 1}$ for $\alpha = 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$.

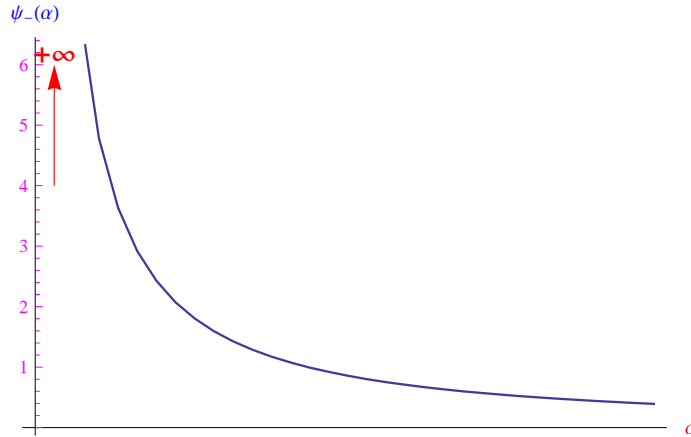
Cioè

$$\psi_-(1) = 1 + \ln 2 - \ln(1 + e) \simeq 0.379885 \quad (31)$$

It follows:

$$\min_{(0,1)} \psi_-(\alpha) = \psi_-(1) = 1 + \ln 2 - \ln(1 + e), \quad \sup_{(0,1)} \psi_-(\alpha) = +\infty$$

In fig. 8 the trend of $\psi_-(\alpha)$.


 Figure 8: Trend of $\psi_-(\alpha)$ obtained numerically.

Proceeding in a similar way for $|I_+(\alpha, \omega)|$

$$|I_+(\alpha, \omega)| \leq \psi_+(\alpha) \quad (32)$$

where

$$\psi_+(\alpha) \stackrel{def}{=} \int_0^{+\infty} \frac{e^{\alpha t}}{e^{e^t} + 1} dt \quad (33)$$

elementarmente esprimibile solo per $\alpha = 1$:

$$\psi_+(1) = \int_0^{+\infty} \frac{e^t}{e^{e^t} + 1} dt \stackrel{x=\ln t}{=} \int_1^{+\infty} \frac{dx}{e^x + 1} = -\ln(1 + e^{-x}) \Big|_{x=1}^{x=+\infty} \quad (34)$$

Cioè

$$\psi_+(1) = \ln(1 + e) - 1 \simeq 0.313262 \quad (35)$$

It is easy to convince oneself that $\psi_+(\alpha)$ is monotonically increasing in $(0, 1)$. So:

$$\begin{aligned} \min_{(0,1)} \psi_+(\alpha) &= \psi_+(0) = \int_0^{+\infty} \frac{dt}{e^{e^t} + 1} \simeq 0.180628 \\ \max_{(0,1)} \psi_+(\alpha) &= \psi_+(1) = \ln(1 + e) - 1 \simeq 0.313262 \end{aligned} \quad (36)$$

The graph is in fig. 9, while in fig. 10 single function graphs are compared. From this analysis

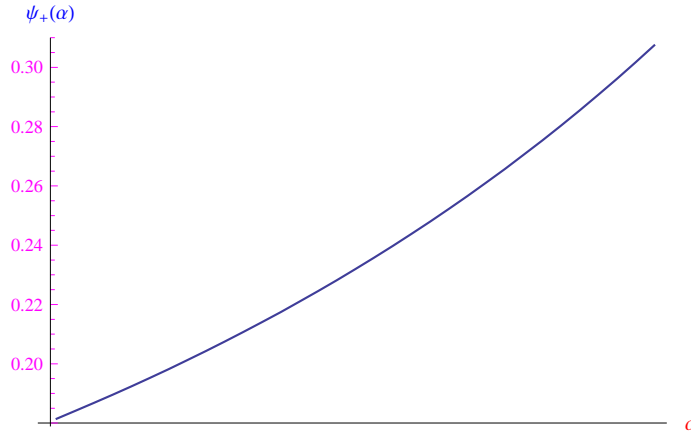


Figure 9: Trend of $\psi_+(\alpha)$ obtained numerically.

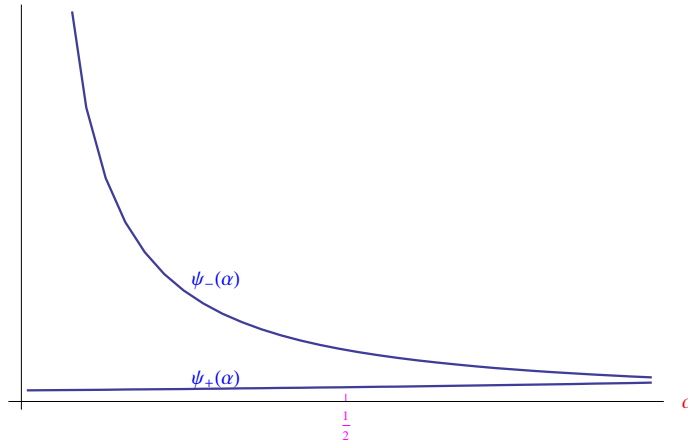


Figure 10: Trend of $\psi_{\pm}(\alpha)$ obtained numerically.

it follows immediately:

$$\alpha \in (0, 1) \implies \psi_-(\alpha) > \psi_+(\alpha) \quad (37)$$

$\forall \alpha \in (0, 1)$ sia $M_* > \psi_+(\frac{1}{2}) > \psi_+(\alpha) > 0$ in corrispondenza del quale esiste $\delta_{M_*} > 0$ tale che

$$0 < \alpha < \delta_{M_*} \implies |I_-(\alpha, \omega)| > M_*$$

a più forte ragione $|I_-(\alpha, \omega)| > |I_+(\alpha, \omega)|$. Ne segue che non esistono zeri per $\alpha \in (0, \delta_{M_*})$ e quindi per la citata proprietà di simmetria della distribuzione degli zeri, non esistono zeri

per $\alpha \in (1 - \delta_{M_*}, 1)$. Ne concludiamo l'esistenza di una striscia critica effettiva:

$$A_{eff} = \{z \in \mathbb{C} \mid \delta_{M_*} < \operatorname{Re} z < 1 - \delta_{M_*}, -\infty < \operatorname{Im} z < +\infty\} \quad (38)$$

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- [3] Smirnov V.I. *Corso di Matematica superiore, vol. II.* Editori Riuniti.