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$$\frac{d}{dx}f\left(x\right) \quad \sum_{k=0}^{+\infty}a_k \quad \int f\left(x\right)dx \quad \oint_{\Gamma}\left(Xdx+Ydy+Zdz\right)$$

Riemann Hypothesis: the effective critical strip

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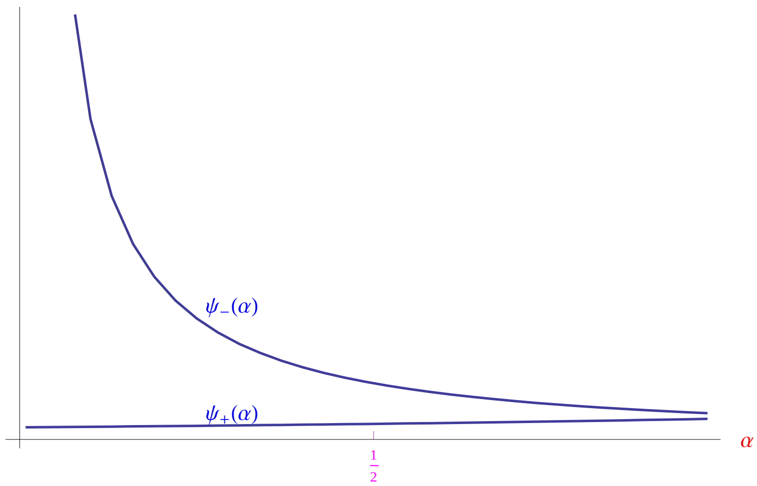

Contents

1 Introduction

The Dirichlet series $\sum_{n=1}^{+\infty} n^{-z}$ converges for $\operatorname{Re} z > 1$, and the sum is the *Riemann zeta function*:

$$\zeta(z) = \sum_{n=1}^{+\infty} \frac{1}{n^z}, \qquad \operatorname{Re} z > 1 \tag{1}$$

Another notable series that can be expressed through the zeta function is:

$$\sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^z} = \left(1 - 2^{1-z}\right) \Gamma(z)\, \zeta(z) \tag{2}$$

which converges for $\operatorname{Re} z > 0$.

Since the series are not very "handy" it is preferable to work with integral representations.

2 A remarkable integral representation

In Quantum Statistical Mechanics the following generalized integrals which are not elementary expressible often appear

$$\int_0^{+\infty} \frac{t^{x-1} dt}{e^t \pm 1} \tag{3}$$

having:

$$\begin{aligned} \int_0^{+\infty} \frac{t^{x-1} dt}{e^t + 1} &= \left(1 - 2^{1-x}\right) \Gamma(x)\, \zeta(x), \quad \forall x \in (0, +\infty) \\ \int_0^{+\infty} \frac{t^{x-1} dt}{e^t - 1} &= \Gamma(x)\, \zeta(x), \quad \forall x \in (1, +\infty) \end{aligned} \tag{4}$$

where $\Gamma(x)$ and $\zeta(x)$ are the Eulerian gamma function and the Riemann zeta function, respectively. Through an elementary change of variable, the first integral becomes

$$\int_{-\infty}^{+\infty} \frac{e^{xt}}{e^{e^t} + 1} dt \tag{5}$$

We define

$$f(x, t) = \frac{e^{xt}}{e^{e^t} + 1}, \quad \begin{cases} x \in (0, 1) & \text{parameter} \\ t \in (-\infty, +\infty) & \text{independent variable} \end{cases} \tag{6}$$

Taking into account the first of (4):

$$\hat{f}(x) \stackrel{def}{=} \int_{-\infty}^{+\infty} f(x, t)\, dt = \left(1 - 2^{1-x}\right) \Gamma(x)\, \zeta(x), \quad \forall x \in (0, +\infty), \tag{7}$$

Proceeding by extension to the complex field, we can define the following function:

$$\hat{f}(z) \equiv \hat{f}(x + iy) = \int_{-\infty}^{+\infty} f(x, t)\, e^{iyt} dt = \left(1 - 2^{1-z}\right) \Gamma(z)\, \zeta(z), \quad \operatorname{Re}(z) > 0 \tag{8}$$

3 The analytic character of $\zeta(z)$, the functional equation and the non-trivial zeros

Riemann found the analytic extension (or *holomorphic extension*) of the sum of the Dirichlet series (1) over all $\mathbb{C}$ except the point $z=1$, which turns out to be a simple pole with residue 1.

The aforesaid analytical extension is represented by the following functional equation [1]:

$$\pi^{-\frac{z}{2}}\Gamma\left(\frac{z}{2}\right)\zeta(z)=\pi^{\frac{z-1}{2}}\Gamma\left(\frac{1-z}{2}\right)\zeta(1-z) \tag{9}$$

The *non-trivial zeros* of $\zeta(z)$ fall in the *critical strip* [1]-[2] of the complex plane defined by

$$A=\{z\in\mathbb{C}\mid 0<\operatorname{Re}z<1,\ \ -\infty<\operatorname{Im}z<+\infty\} \tag{10}$$

Proposition 1

$$\left|\left(1-2^{1-z}\right)\Gamma(z)\right|>0,\quad \forall z\in A \tag{11}$$

$$z_0\in A\mid \zeta(z_0)=0\Longleftrightarrow\zeta(1-z_0)=0 \tag{12}$$

Proof. The inequality (11) derives from the fact that the gamma function has no zeros [3], while $1-2^{1-z}$ is manifestly zero-free in A.

(12) is a consequence of the functional equation (9). ■

From the proposition just proved it follows $f(z)$ and $\zeta(z)$ have the same (non-trivial) zeros. Dalla (12) segue che gli zeri sono simmetrici rispetto alla retta $\operatorname{Re}s=1/2$. Furthermore, it can be observed that $\zeta(z^*)=\zeta(z)^*$ where $*$ denotes the complex conjugate. This implies that the nontrivial zeros are symmetric about the real axis (see fig. 1).

The line $\operatorname{Re}s=1/2$ is called the *critical line*. Hardy [1]-[2] proved that infinitely many zeros fall on this line.

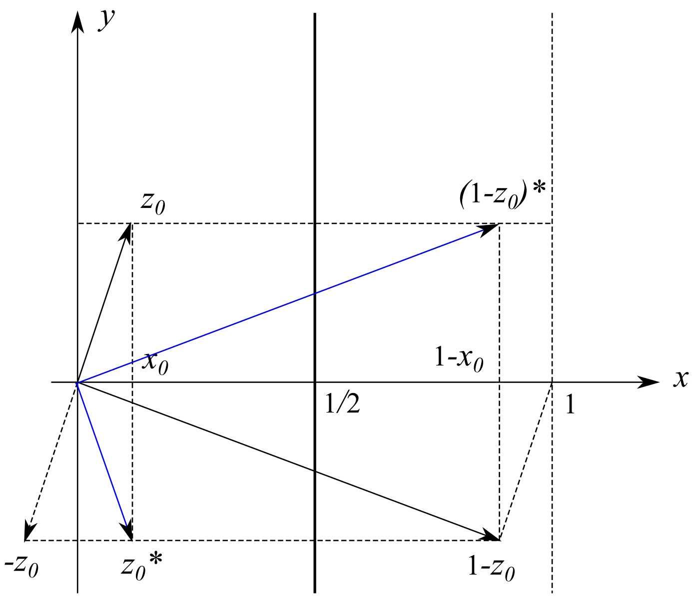

Figure 1: Symmetry of the distribution of non trivial zeros.

4 Riemann Hypothesis

4.1 Fourier Transform

From (8) we see that for a given $x \in (0,1)$ the complex function $f(x+iy)$ is the Fourier transform of (6).

Conjecture 2 *(**Riemann Hypothesis**)*
The non-trivial zeros of the function

$$\hat{f}(x+iy) = \int_{-\infty}^{+\infty} f(x,t)\, e^{iyt} dt \tag{13}$$

have real part $x = 1/2$.

Let us first study the behavior of the function $f(x,t)$ (given by (6)) which for each value of the parameter $x \in (0,1)$ is defined in $(-\infty,+\infty)$.

Sign and intersections with the axes

It turns out $g(x,t) > 0$, $\forall t \in (-\infty,+\infty)$ for which the graph of f lies in the semi-plane of the positive ordinates. It does not intersect the abscissa axis, while it does intersect the ordinate axis at $\left(0,(e+1)^{-1}\right)$.

Behavior at extremes

After calculations:

$$\lim_{t\to+\infty} f(x,t) = 0^+, \quad \forall x \in (0,1)$$

The order of infinitesimal:

$$\lim_{t\to+\infty} t^{\alpha} f(x,t) = 0^+, \quad \forall \alpha > 0 \qquad \text{(infinitesimal of infinitely large order)} \tag{14}$$

$$\lim_{t\to-\infty} f(x,t) = \begin{cases} \frac{1}{2}^-, & \text{if } x = 0 \\ 0^+, & \text{if } x > 0 \end{cases}$$

Precisely:

$$\lim_{t\to-\infty} t^{\alpha} f(x>0,t) = 0^+, \quad \forall \alpha > 0 \tag{15}$$

Conclusion: for $|t| \to +\infty$ the function $f(x>0,t)$ is an infinitesimal of order infinitely large, provided that it is $x > 0$.

First derivative

$$f'(x,t) \equiv \frac{\partial}{\partial t} f(x,t) = \frac{e^{xt}\left[x\left(e^{e^t}+1\right) - e^{t+e^t}\right]}{\left(e^{e^t}+1\right)^2}$$

For $x = 0$

$$f'(0,t) = -\frac{e^{t+e^t}}{\left(e^{e^t}+1\right)^2} < 0, \quad \forall t \in (-\infty,+\infty)$$

so the function is strictly decreasing.

For $x > 0$

$$f'(x,t) = 0 \iff x\left(e^{e^t}+1\right) - e^{t+e^t} = 0 \tag{16}$$

which is solved numerically. After calculations, the root of the (16) è

$$0 < x < 1 \Longrightarrow t_*(x) \in [\sim -6.32, 0.2]$$

Some values for assigned $x \in (0,1)$:

$$t_* \left(\frac{1}{5}\right) \simeq -1.07$$
$$t_* \left(\frac{1}{4}\right) \simeq -0.88$$
$$t_* \left(\frac{1}{2}\right) \simeq -0.30$$
$$t_* \left(\frac{2}{3}\right) \simeq -0.07$$
$$t_* \left(\frac{3}{4}\right) \simeq 0.02$$

The sign is

$$-\infty < t < t_* (x) \Longrightarrow f' (x,t) > 0$$
$$t_* (x) < t < +\infty \Longrightarrow f' (x,t) < 0$$

Hence the function is strictly increasing in $(-\infty, t_* (x))$ is strictly decreasing in $(t_* (x), +\infty)$. So $t_* (x)$ is a point of relative maximum for

Second derivative

$$f'' (x,t) = \frac{e^{xt} \left[e^{2\left(e^t+t\right)} - e^{e^t+2t} + x^2 \left(1+e^{e^t}\right)^2 - (2x+1)\left(e^{t+e^t} + e^{2e^t+t}\right)\right]}{\left(1+e^{e^t}\right)^3} \tag{17}$$

For $x = 0$

$$f'' (0,t) = \frac{e^{2\left(e^t+t\right)} - e^{e^t+2t} - \left(e^{t+e^t} + e^{2e^t+t}\right)}{\left(1+e^{e^t}\right)^3}$$

which has a zero in $t'_* (x=0) \simeq 0.43$. The sign is

$$-\infty < t < t'_* (x=0) \Longrightarrow f'' (0,t) < 0$$
$$t'_* (x=0) < t < +\infty \Longrightarrow f'' (0,t) > 0$$

It follows that the graph of $f(0,t)$ is convex in $(-\infty, t'_* (x=0))$ and concave in $(t'_* (x=0), +\infty)$. So $(0.43, 0.18)$ is an inflection point with an oblique tangent. In fig. 2 we report the graph of $f(0,t)$.

For $x > 0$ we perform a qualitative analysis. The parameter x controls the slope of the graph of $f(t)$ in $(-\infty, 0)$ since

$$\frac{\partial}{\partial t} e^{xt} = x e^{xt}$$

For $t \in (0, +\infty)$ the slope is controlled by e^{e^t} in denominator. This implies that the effects of the parameter x are felt for $t \in (-\infty, 0)$, while in $(0, +\infty)$ the trend is practically independent of this parameter. Fig. 3 plots $f(x,t)$ for increasing values of the parameter x starting from $x = 0$.

We rewrite (7)

$$F(x) = \int_{-\infty}^{+\infty} f(x,t)\, dt \tag{18}$$

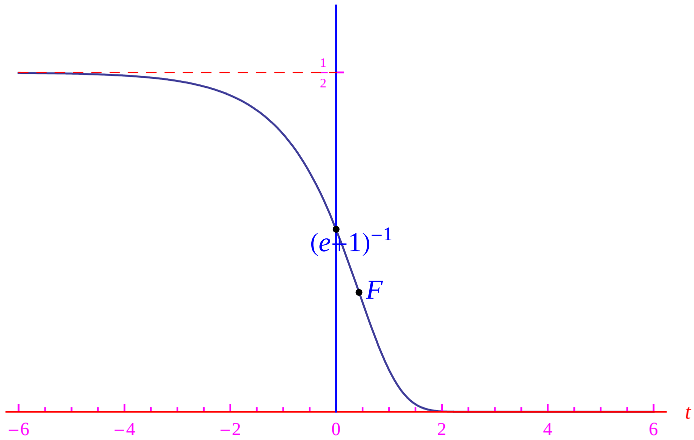

Figure 2: Trend of $f(0,t)$.

which *Mathematica* calculates through

$$F(x) = \left(1-2^{1-x}\right)\Gamma(x)\zeta(x)$$

As previously seen, for $x>0$ the integrand function is for $t\to\pm\infty$ an infinitesimal of infinitely large order; so the integral converges. More precisely:

$$F(x) = \int_{-\infty}^{0}\frac{e^{xt}}{e^{e^t}+1}dt + \underbrace{\int_{0}^{+\infty}\frac{e^{xt}}{e^{e^t}+1}dt}_{\text{converges }\forall x\in(0,1)}$$

For $x=0$

$$f(0,t) = \frac{1}{e^{e^t}+1}\underset{t\to-\infty}{\longrightarrow}\frac{1}{2}\Longrightarrow\int_{-\infty}^{0}\frac{dt}{e^{e^t}+1} = +\infty\Longrightarrow\lim_{x\to0^+}F(x) = +\infty$$

For $x>0$ the trend in $(-\infty,0)$ is dominated by e^{xt}

$$\frac{e^{xt}}{e^{e^t}+1}\underset{t\to-\infty}{\longrightarrow}e^{xt}$$

so the integral converges. As x increases in $(-\infty,0)$ the slope increases, and this favors the convergence of the integral[1], simultaneously decreases the area of the base trapezoid $(-\infty,0)$ and therefore the value of $F(x)$. This shows that $G(x)$ is strictly decreasing, as confirmed by the graph fig. 4 obtained with *Mathematica*.

[1]The parameter x therefore controls the speed of convergence of the integral in the interval $(-\infty,0)$.

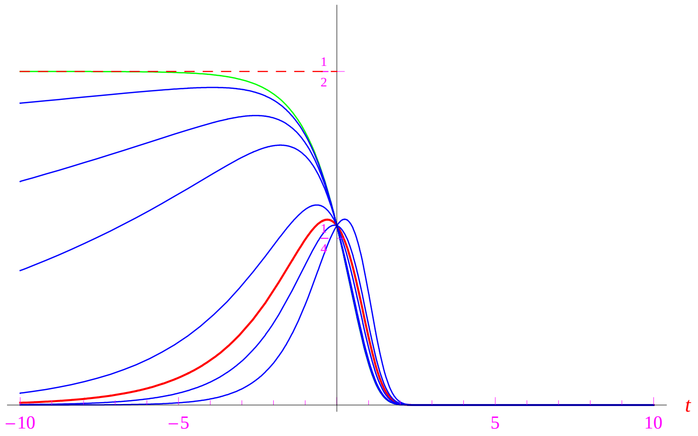

Figure 3: Trend of $f(x,t)$ for different values of x. Curve in green: $x=0$. The flattest curve towards the ordinate axis is for $x=1$.

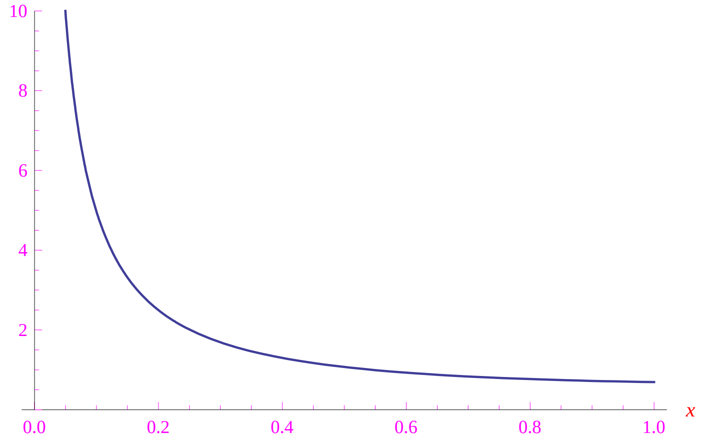

Figure 4: Trend of $F(x)$.

4.2 Zeros of the Fourier Transform

We redefine the variables x, y in α, ω, and then rewrite (13):

$$I(\alpha,\omega) = \int_{-\infty}^{+\infty} \frac{e^{\alpha t}}{e^{e^t}+1} e^{i\omega t} dt \tag{19}$$

It follows

$$\begin{aligned} I(\alpha,\omega) &= I_-(\alpha,\omega) + I_+(\alpha,\omega) \\ I_-(\alpha,\omega) &\stackrel{def}{=} \int_{-\infty}^{0} \frac{e^{\alpha t}}{e^{e^t}+1} e^{i\omega t} dt \\ I_+(\alpha,\omega) &\stackrel{def}{=} \int_{0}^{+\infty} \frac{e^{\alpha t}}{e^{e^t}+1} e^{i\omega t} dt \end{aligned} \tag{20}$$

Even if we are interested in $\alpha \in (0,1)$, it results:

$$|I_-(\alpha,\omega)| = \int_{-\infty}^{0} \frac{e^{\alpha t}}{e^{e^t}+1} dt = +\infty, \quad \forall \alpha \in (-\infty, 0]; \quad \left|I_\alpha^{(-)}(\omega)\right| < +\infty, \ \forall \alpha \in (0,+\infty)$$
$$|I_+(\alpha,\omega)| < +\infty, \quad \forall \alpha \in (-\infty,+\infty)$$

Conclusion 3 *The variable α conditions the convergence of $I_-(\alpha,\omega)$ but not that of $I_+(\alpha,\omega)$.*

The integral (19) can be seen as:

- complex function of the real variables (α,ω);
- complex function of the complex variable $\alpha + i\omega$;
- family of functions of ω, with one real parameter α.

Due to the symmetry property established in the number 3, we can limit the search for zeros in the region:

$$A_1 = \left\{ (\alpha,\omega) \in \mathbb{R}^2 \mid 0 < \alpha < \frac{1}{2}, \ 0 \leq \omega < +\infty \right\} \tag{21}$$

It follows

$$I(\alpha,\omega) = 0 \Longleftrightarrow I_-(\alpha,\omega) = -I_+(\alpha,\omega) \Longrightarrow |I_-(\alpha,\omega)| = |I_+(\alpha,\omega)| \tag{22}$$

We study the behavior of the individual modules for $\alpha \in (0,1)$.

$$|I_-(\alpha,\omega)| = \left| \int_{-\infty}^{0} \frac{e^{\alpha t}}{e^{e^t}+1} e^{i\omega t} dt \right| \leq \int_{-\infty}^{0} \left| \frac{e^{\alpha t}}{e^{e^t}+1} e^{i\omega t} \right| dt = \int_{-\infty}^{0} \frac{e^{\alpha t}}{e^{e^t}+1} dt,$$

i.e.

$$|I_-(\alpha,\omega)| \leq \psi_-(\alpha) \tag{23}$$

where

$$\psi_-(\alpha) \stackrel{def}{=} \int_{-\infty}^{0} \frac{e^{\alpha t}}{e^{e^t}+1} dt \tag{24}$$

We have

$$\sup_{\mathbb{R}}\left(\frac{1}{e^{e^t}+1}\right)=\frac{1}{2}\Longrightarrow\int_{-\infty}^{0}\frac{e^{\alpha t}}{e^{e^t}+1}dt<\frac{1}{2}\int_{-\infty}^{0}e^{\alpha t}dt=\frac{1}{2\alpha}$$

So

$$|I_-(\alpha,\omega)|<\frac{1}{2\alpha},\quad\forall\alpha\in(0,1)\tag{25}$$

Incidentally

$$\lim_{\alpha\to 0^+}|I_-(\alpha,\omega)|=\lim_{\alpha\to 0^+}\frac{1}{2\alpha}=+\infty\tag{26}$$

From (25): for $\alpha\to 0^+$ and $\forall\omega\in\mathbb{R}$, the function $\left|I_\alpha^{(-)}(\omega)\right|$ is an infinitesimal of order $\beta<1$ (assuming α^{-1} as the reference infinitesimal). Furthermore:

$$\forall M>0,\ \exists\Delta_M(\omega)>0\mid 0<\alpha<\Delta_M(\omega)\Longrightarrow|I_-(\alpha,\omega)|>M\tag{27}$$

Let's consider

$$\delta_M=\inf_{\mathbb{R}}\{\Delta_M(\omega)\}>0\tag{28}$$

$$\forall M>0,\ \exists\delta_M>0\mid 0<\alpha<\delta_M\Longrightarrow|I_-(\alpha,\omega)|>M,\quad\forall\omega\in\mathbb{R}\tag{29}$$

The function (24) is monotonically decreasing in $(0,1)$. This can be deduced from the trend of the integrand function for different values of $\alpha\in(0,1)$ (fig. 5).

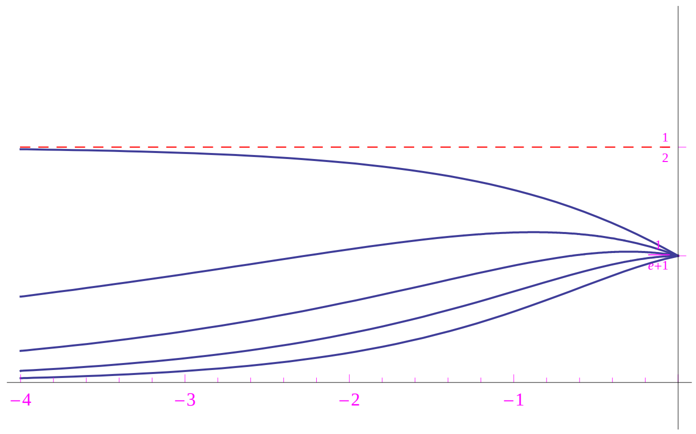

Figure 5: Trend of $\frac{\varepsilon^{\alpha t}}{e^{e^t}+1}$ for $\alpha=0,\frac{1}{4},\frac{1}{2},\frac{3}{4},1$.

Proceeding in a similar way for $|I_+(\alpha,\omega)|$

$$|I_+(\alpha,\omega)|\leq\psi_+(\alpha)\tag{30}$$

where

$$\psi_+(\alpha)\stackrel{def}{=}\int_0^{+\infty}\frac{e^{\alpha t}}{e^{e^t}+1}dt\tag{31}$$

The functions $\psi_\pm(\alpha)$ are not elementarily expressible, except for $\alpha=1$. Precisely:

$$\psi_-(1)=\int_{-\infty}^{0}\frac{dt}{e^{e^t}+1}=1+\ln 2-\ln(1+e)\tag{32}$$

$$\psi_+(1)=\ln(1+e)-1<\psi_-(1)$$

It follows:

$$\min_{(0,1)} \psi_-(\alpha) = \psi_-(1) = 1 + \ln 2 - \ln(1+e), \quad \sup_{(0,1)} \psi_-(\alpha) = +\infty$$

In fig. 6 the trend of $\psi_-(\alpha)$.

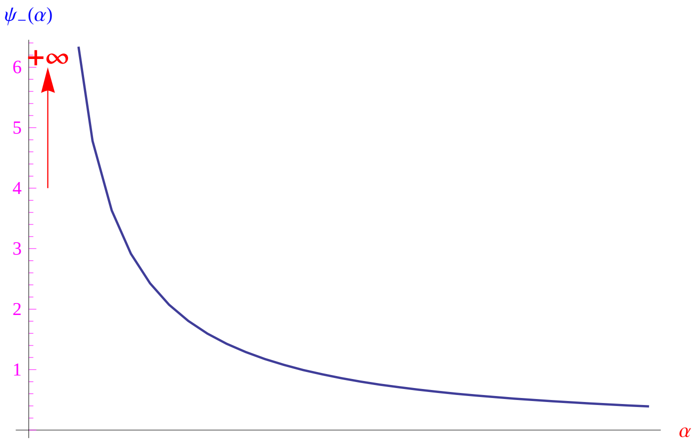

Figure 6: Trend of $\psi_-(\alpha)$ obtained numerically.

It is easy to convince oneself that $\psi_+(\alpha)$ is monotonically increasing in $(0,1)$. So:

$$\min_{(0,1)} \psi_+(\alpha) = \psi_+(0) = \int_0^{+\infty} \frac{dt}{e^{e^t}+1} \simeq 0.180628 \tag{33}$$
$$\max_{(0,1)} \psi_+(\alpha) = \psi_+(1) = \ln(1+e) - 1$$

The graph is in fig. 7, while in fig. 8 single function graphs are compared.

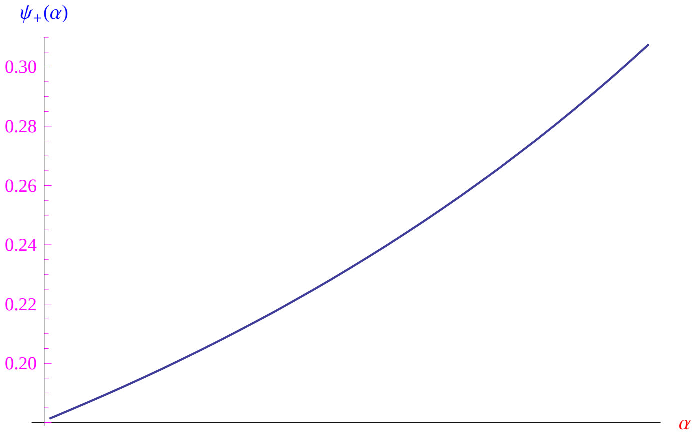

Figure 7: Trend of $\psi_+(\alpha)$ obtained numerically.

From this analysis it follows immediately:

$$\alpha \in (0,1) \Longrightarrow \psi_-(\alpha) > \psi_+(\alpha) \tag{34}$$

It can manifestly happen:

$$|I_-(\alpha_0,\omega_0)| = |I_+(\alpha_0,\omega_0)| \leq \psi_+(\alpha_0) < \psi_-(\alpha_0) \tag{35}$$

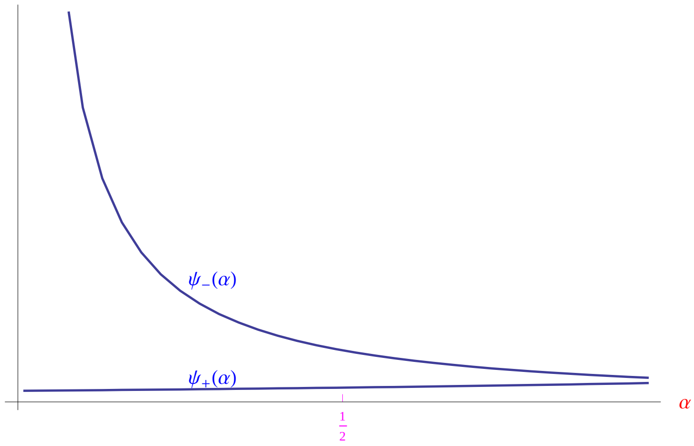

Figure 8: Trend of $\psi_\pm(\alpha)$ obtained numerically.

compatibly with (34). In other words, (34) does not rule out the presence of zeros in A_1. It follows that a sufficient condition for the non-existence of zeros in A_1 is

$$|I_-(\alpha,\omega)| > |I_+(\alpha,\omega)|, \quad \forall(\alpha,\omega)\in A_1 \tag{36}$$

so that the right chains of inequalities are respectively:

$$\begin{aligned} &|I_+(\alpha,\omega)| \leq \psi_+(\alpha_0) < |I_-(\alpha,\omega)| \leq \psi_-(\alpha_0) \\ &|I_+(\alpha,\omega)| < |I_-(\alpha,\omega)| \leq \psi_+(\alpha_0) < \psi_-(\alpha_0) \end{aligned} \tag{37}$$

From (29) taking $M_* > \psi_+\left(\frac{1}{2}\right)$, we have

$$\exists\delta_{M_*} > 0 \mid 0 < \alpha < \delta_M \Longrightarrow |I_-(\alpha,\omega)| > M_* > \psi_+\left(\frac{1}{2}\right) > \psi_+(\alpha), \ \forall\alpha\in\left(0,\frac{1}{2}\right), \quad \forall\omega\in\mathbb{R}$$

per cui non esistono zeri in $(0,\delta_{M_*})$ e per le note proprietà di simmetria, in $(1-\delta_{M_*},1)$. Ne segue che la striscia critica effettiva è (fig. 9)

$$A_{eff} = \{z\in\mathbb{C} \mid \delta_{M_*} < \operatorname{Re} z < 1-\delta_{M_*}, \ \ -\infty < \operatorname{Im} z < +\infty\} \tag{38}$$

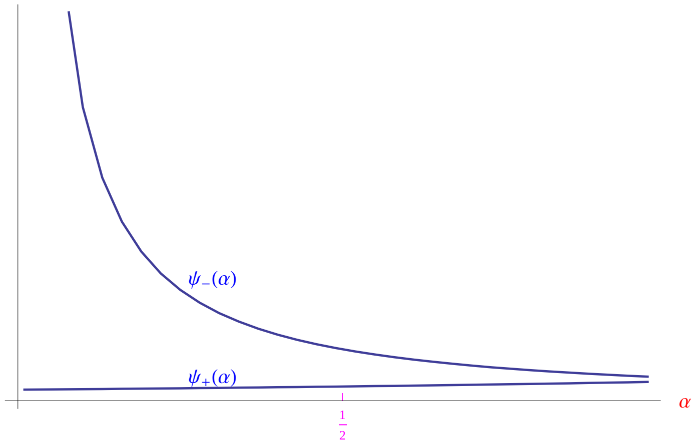

Figure 9: The effective critical strip.

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[3] Smirnov V.I. *Lezioni di Analisi Matematica, vol. II.* Editori Riuniti.