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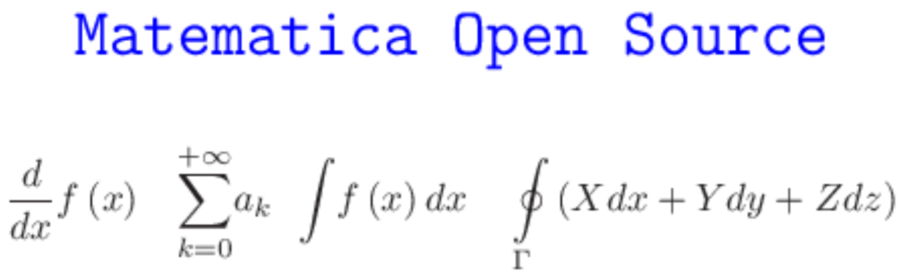
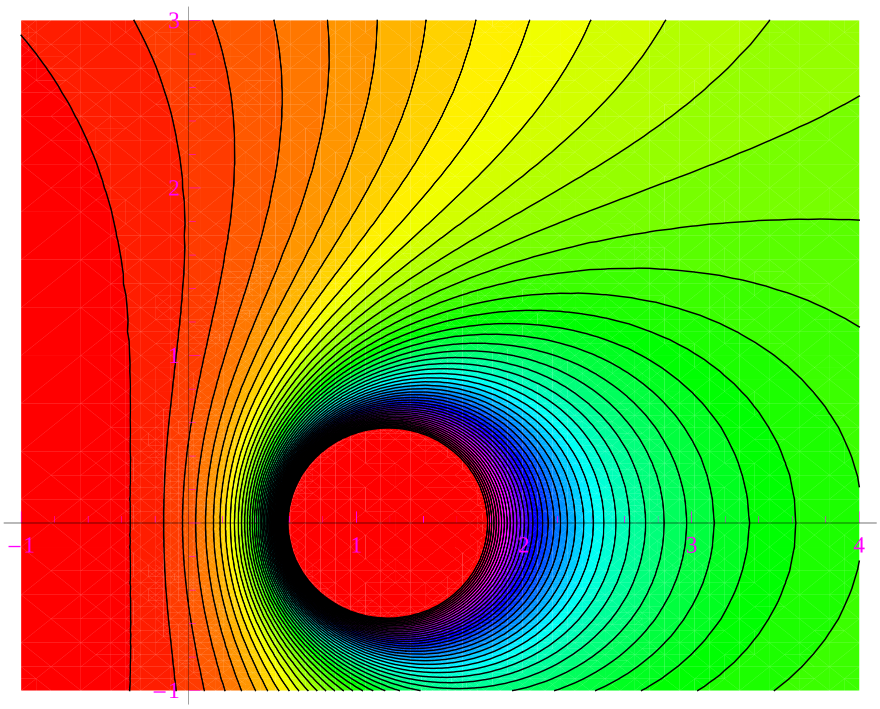


Where are the zeros of the Riemann zeta function?

Marcello Colozzo



Contents

1 Introduction

The Dirichlet series $\sum_{n=1}^{+\infty} n^{-z}$ converges for $\operatorname{Re} z > 1$, and the sum is the *Riemann zeta function*:

$$\zeta(z) = \sum_{n=1}^{+\infty} \frac{1}{n^z}, \quad \operatorname{Re} z > 1 \tag{1}$$

Another notable series that can be expressed through the zeta function is:

$$\sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^z} = \left(1 - 2^{1-z}\right) \Gamma(z) \zeta(z)$$

which converges for $\operatorname{Re} z > 0$.

Since the series are not very "handy" it is preferable to work with integral representations.

2 A remarkable integral representation

In Quantum Statistical Mechanics the following generalized integrals which are not elementary expressible often appear

$$\int_0^{+\infty} \frac{t^{x-1} dt}{e^t \pm 1} \tag{2}$$

having:

$$\begin{aligned} \int_0^{+\infty} \frac{t^{x-1} dt}{e^t + 1} &= \left(1 - 2^{1-x}\right) \Gamma(x) \zeta(x), \quad \forall x \in (0, +\infty) \\ \int_0^{+\infty} \frac{t^{x-1} dt}{e^t - 1} &= \Gamma(x) \zeta(x), \quad \forall x \in (1, +\infty) \end{aligned} \tag{3}$$

where $\Gamma(x)$ and $\zeta(x)$ are the Eulerian gamma function and the Riemann zeta function, respectively. Through an elementary change of variable, the first integral becomes

$$\int_{-\infty}^{+\infty} \frac{e^{xt}}{e^{e^t} + 1} dt \tag{4}$$

We define

$$f(x,t) = \frac{e^{xt}}{e^{e^t} + 1}, \quad \begin{cases} x \in (0,1) & \text{parameter} \\ t \in (-\infty, +\infty) & \text{independent variable} \end{cases} \tag{5}$$

Taking into account the first of (3):

$$\hat{f}(x) \stackrel{def}{=} \int_{-\infty}^{+\infty} f(x,t)\, dt = \left(1 - 2^{1-x}\right) \Gamma(x) \zeta(x), \quad \forall x \in (0, +\infty), \tag{6}$$

Proceeding by extension to the complex field, we can define the following function:

$$\hat{f}(z) \equiv \hat{f}(x + iy) = \int_{-\infty}^{+\infty} f(x,t)\, e^{iyt} dt = \left(1 - 2^{1-z}\right) \Gamma(z) \zeta(z), \quad \operatorname{Re}(z) > 0 \tag{7}$$

The *non-trivial zeros* of $\zeta(z)$ fall in the *critical strip* [1] of the complex plane defined by

$$A = \{ z \in \mathbb{C} \mid 0 < \operatorname{Re} z < 1, \ -\infty < \operatorname{Im} z < +\infty \} \tag{8}$$

Proposition 1

$$\left|\left(1-2^{1-z}\right)\Gamma\left(z\right)\right|>0,\quad \forall z\in A \tag{9}$$

Proof. The inequality (9) derives from the fact that the gamma function has no zeros [2], while $1-2^{1-z}$ is manifestly zero-free in A. ∎

From the proposition just proved it follows $f\left(z\right)$ and $\zeta\left(z\right)$ have the same (non-trivial) zeros.

3 Riemann Hypothesis

3.1 Fourier Transform

From (7) we see that for a given $x \in (0,1)$ the complex function $f(x+iy)$ is the Fourier transform of (5).

Conjecture 2 *(**Riemann Hypothesis**)*
The non-trivial zeros of the function

$$\hat{f}(x+iy) = \int_{-\infty}^{+\infty} f(x,t)\, e^{iyt} dt \tag{10}$$

have real part $x = 1/2$.

Let us first study the behavior of the function $f(x,t)$ (given by (5)) which for each value of the parameter $x \in (0,1)$ is defined in $(-\infty, +\infty)$.

Sign and intersections with the axes

It turns out $g(x,t) > 0$, $\forall t \in (-\infty, +\infty)$ for which the graph of f lies in the semi-plane of the positive ordinates. It does not intersect the abscissa axis, while it does intersect the ordinate axis at $\left(0, (e+1)^{-1}\right)$.

Behavior at extremes

After calculations:

$$\lim_{t \to +\infty} f(x,t) = 0^+, \quad \forall x \in (0,1)$$

The order of infinitesimal:

$$\lim_{t \to +\infty} t^\alpha f(x,t) = 0^+, \quad \forall \alpha > 0 \qquad \text{(infinitesimal of infinitely large order)} \tag{11}$$

$$\lim_{t \to -\infty} f(x,t) = \begin{cases} \frac{1}{2}^-, & \text{if } x = 0 \\ 0^+, & \text{if } x > 0 \end{cases}$$

Precisely:

$$\lim_{t \to -\infty} t^\alpha f(x > 0, t) = 0^+, \quad \forall \alpha > 0 \tag{12}$$

Conclusion: for $|t| \to +\infty$ the function $f(x > 0, t)$ is an infinitesimal of order infinitely large, provided that it is $x > 0$.

First derivative

$$f'(x,t) \equiv \frac{\partial}{\partial t} f(x,t) = \frac{e^{xt}\left[x\left(e^{e^t}+1\right) - e^{t+e^t}\right]}{\left(e^{e^t}+1\right)^2}$$

For $x = 0$

$$f'(0,t) = -\frac{e^{t+e^t}}{\left(e^{e^t}+1\right)^2} < 0, \quad \forall t \in (-\infty, +\infty)$$

so the function is strictly decreasing.

For $x > 0$

$$f'(x,t) = 0 \Longleftrightarrow x\left(e^{e^t}+1\right) - e^{t+e^t} = 0 \tag{13}$$

which is solved numerically. After calculations, the root of the (13) è

$$0 < x < 1 \Longrightarrow t_*(x) \in [\sim -6.32, 0.2]$$

Some values for assigned $x \in (0,1)$:

$$t_{*}\left(\frac{1}{5}\right) \simeq -1.07$$
$$t_{*}\left(\frac{1}{4}\right) \simeq -0.88$$
$$t_{*}\left(\frac{1}{2}\right) \simeq -0.30$$
$$t_{*}\left(\frac{2}{3}\right) \simeq -0.07$$
$$t_{*}\left(\frac{3}{4}\right) \simeq 0.02$$

The sign is

$$-\infty < t < t_{*}(x) \Longrightarrow f'(x,t) > 0$$
$$t_{*}(x) < t < +\infty \Longrightarrow f'(x,t) < 0$$

Hence the function is strictly increasing in $(-\infty, t_{*}(x))$ is strictly decreasing in $(t_{*}(x), +\infty)$. So $t_{*}(x)$ is a point of relative maximum for

Second derivative

$$f''(x,t) = \frac{e^{xt}\left[e^{2\left(e^{t}+t\right)} - e^{e^{t}+2t} + x^{2}\left(1+e^{e^{t}}\right)^{2} - (2x+1)\left(e^{t+e^{t}} + e^{2e^{t}+t}\right)\right]}{\left(1+e^{e^{t}}\right)^{3}} \tag{14}$$

For $x = 0$

$$f''(0,t) = \frac{e^{2\left(e^{t}+t\right)} - e^{e^{t}+2t} - \left(e^{t+e^{t}} + e^{2e^{t}+t}\right)}{\left(1+e^{e^{t}}\right)^{3}}$$

which has a zero in $t'_{*}(x=0) \simeq 0.43$. The sign is

$$-\infty < t < t'_{*}(x=0) \Longrightarrow f''(0,t) < 0$$
$$t'_{*}(x=0) < t < +\infty \Longrightarrow f''(0,t) > 0$$

It follows that the graph of $f(0,t)$ is convex in $(-\infty, t'_{*}(x=0))$ and concave in $(t'_{*}(x=0), +\infty)$. So $(0.43, 0.18)$ is an inflection point with an oblique tangent. In fig. 1 we report the graph of $f(0,t)$.

For $x > 0$ we perform a qualitative analysis. The parameter x controls the slope of the graph of $f(t)$ in $(-\infty, 0)$ since

$$\frac{\partial}{\partial t} e^{xt} = x e^{xt}$$

For $t \in (0, +\infty)$ the slope is controlled by $e^{e^{t}}$ in denominator. This implies that the effects of the parameter x are felt for $t \in (-\infty, 0)$, while in $(0, +\infty)$ the trend is practically independent of this parameter. Fig. 2 plots $f(x,t)$ for increasing values of the parameter x starting from $x = 0$.

We rewrite (6)

$$F(x) = \int_{-\infty}^{+\infty} f(x,t)\, dt \tag{15}$$

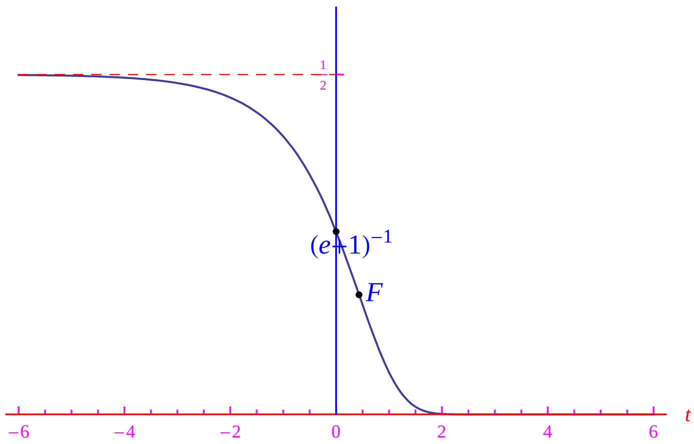


Figure 1: Trend of $f(0,t)$.

which *Mathematica* calculates through

$$F(x) = \left(1 - 2^{1-x}\right)\Gamma(x)\,\zeta(x)$$

As previously seen, for $x > 0$ the integrand function is for $t \to \pm\infty$ an infinitesimal of infinitely large order; so the integral converges. More precisely:

$$F(x) = \int_{-\infty}^{0} \frac{e^{xt}}{e^{e^t}+1}dt + \underbrace{\int_{0}^{+\infty} \frac{e^{xt}}{e^{e^t}+1}dt}_{\text{converges } \forall x \in (0,1)}$$

For $x = 0$

$$f(0,t) = \frac{1}{e^{e^t}+1} \underset{t\to-\infty}{\longrightarrow} \frac{1}{2} \Longrightarrow \int_{-\infty}^{0} \frac{dt}{e^{e^t}+1} = +\infty \Longrightarrow \lim_{x\to 0^+} F(x) = +\infty$$

For $x > 0$ the trend in $(-\infty, 0)$ is dominated by e^{xt}

$$\frac{e^{xt}}{e^{e^t}+1} \underset{t\to-\infty}{\longrightarrow} e^{xt}$$

so the integral converges. As x increases in $(-\infty, 0)$ the slope increases, and this favors the convergence of the integral[1], simultaneously decreases the area of the base trapezoid $(-\infty, 0)$ and therefore the value of $F(x)$. This shows that $G(x)$ is strictly decreasing, as confirmed by the graph fig. 3 obtained with *Mathematica*.

[1]The parameter x therefore controls the speed of convergence of the integral in the interval $(-\infty, 0)$.

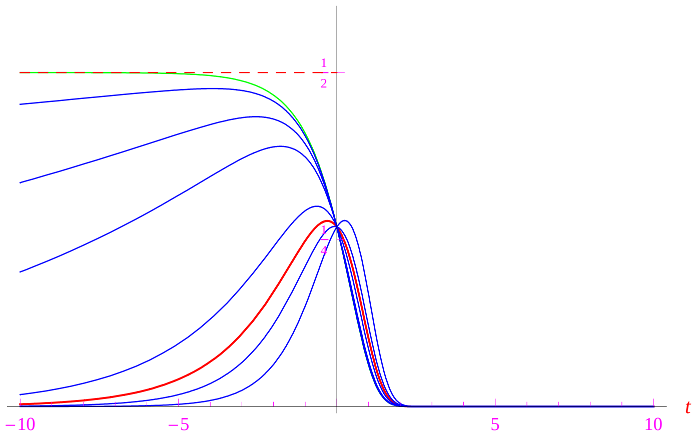


Figure 2: Trend of $f(x,t)$ for different values of x. Curve in green: $x=0$. The flattest curve towards the ordinate axis is for $x=1$.

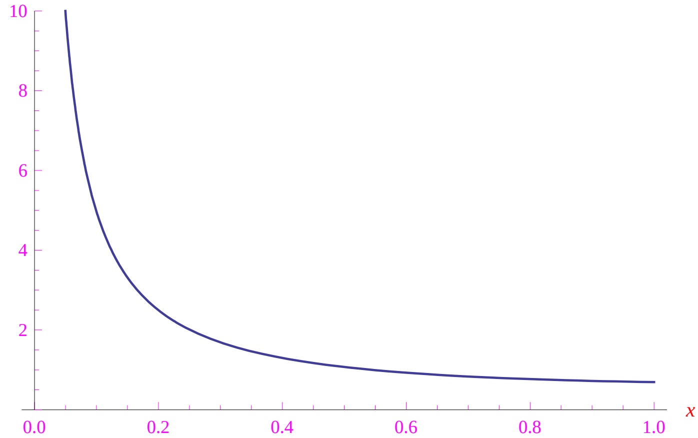


Figure 3: Trend of $F(x)$.

3.2 Zeros of the Fourier Transform

Ridefiniamo le variabili x, y in α, ω. Quindi la (10) diviene:

$$\hat{f}_\alpha(\omega) = \int_{-\infty}^{+\infty} \frac{e^{\alpha t}}{e^{e^t}+1} e^{i\omega t} dt \tag{16}$$

Segue

$$\begin{aligned} \hat{f}_\alpha(\omega) = \; & I_\alpha^{(-)}(\omega) + I_\alpha^{(+)}(\omega) \stackrel{def}{=} I_\alpha(\omega) \\ I_\alpha^{(-)}(\omega) & \stackrel{def}{=} \int_{-\infty}^{0} \frac{e^{\alpha t}}{e^{e^t}+1} e^{i\omega t} dt \\ I_\alpha^{(+)}(\omega) & \stackrel{def}{=} \int_{0}^{+\infty} \frac{e^{\alpha t}}{e^{e^t}+1} e^{i\omega t} dt \end{aligned} \tag{17}$$

Anche se siamo interessati a $\alpha \in (0,1)$, risulta:

$$\left|I_\alpha^{(-)}(\omega)\right| = \int_{-\infty}^{0} \frac{e^{\alpha t}}{e^{e^t}+1} dt = +\infty, \quad \forall \alpha \in (-\infty, 0]; \quad \left|I_\alpha^{(-)}(\omega)\right| < +\infty, \ \forall \alpha \in (0, +\infty)$$
$$\left|I_\alpha^{(+)}(\omega)\right| < +\infty, \quad \forall \alpha \in (-\infty, +\infty)$$

Conclusion 3 *La convergenza di* $\int_{-\infty}^{+\infty} \frac{e^{\alpha t}}{e^{e^t}+1} e^{i\omega t} dt$ *è condizionata da* α *e non da* ω*. Inoltre,* α *condiziona la convergenza di* $I_\alpha^{(-)}(\omega)$ *ma non quella di* $I_\alpha^{(+)}(\omega)$*.*

Studiamo il comportamento dei singoli moduli $\left|I_\alpha^{(\pm)}(\omega)\right|$ al variare di $\alpha \in (0,1)$.

$$\left|I_\alpha^{(-)}(\omega)\right| = \left|\int_{-\infty}^{0} \frac{e^{\alpha t}}{e^{e^t}+1} e^{i\omega t} dt\right| \leq \int_{-\infty}^{0} \left|\frac{e^{\alpha t}}{e^{e^t}+1} e^{i\omega t}\right| dt = \int_{-\infty}^{0} \frac{e^{\alpha t}}{e^{e^t}+1} dt$$

Cioè

$$\left|I_\alpha^{(-)}(\omega)\right| \leq \int_{-\infty}^{0} \frac{e^{\alpha t}}{e^{e^t}+1} dt \tag{18}$$

L'integrale a secondo membro può essere a sua volta maggiorato, giacché $\sup\left(\left(e^{e^t}+1\right)^{-1}\right) = 1/2$:

$$\int_{-\infty}^{0} \frac{e^{\alpha t}}{e^{e^t}+1} dt < \frac{1}{2} \int_{-\infty}^{0} e^{\alpha t} dt = \frac{1}{2\alpha}$$

onde

$$\left|I_\alpha^{(-)}(\omega)\right| < \frac{1}{2\alpha}, \quad \forall \alpha \in (0,1) \tag{19}$$

Notation 4 *Questo procedimento consente di svincolarci da* ω*, poiché passando ai moduli si ha* $|e^{i\omega t}| = 1$*.*

Incidentalmente

$$\lim_{\alpha \to 0^+} \left|I_\alpha^{(-)}(\omega)\right| = \lim_{\alpha \to 0^+} \frac{1}{2\alpha} = +\infty \tag{20}$$

Segue

$$\forall \varepsilon > 0, \ \exists \delta_\varepsilon > 0 \mid 0 < \alpha < \delta_\varepsilon \Longrightarrow \left|I_\alpha^{(-)}(\omega)\right| > \varepsilon$$

Dalla (19): per $\alpha \to 0^+$, la funzione $\left|I_\alpha^{(-)}(\omega)\right|$ è un infinitesimo di ordine $\beta < 1$ (assumendo come infinitesimo di riferimento α^{-1}). Quindi (fig. 4):

$$\left|I_\alpha^{(-)}(\omega)\right| \simeq \frac{1}{\alpha^\beta}, \quad \alpha \in (0, \delta_\varepsilon) \tag{21}$$

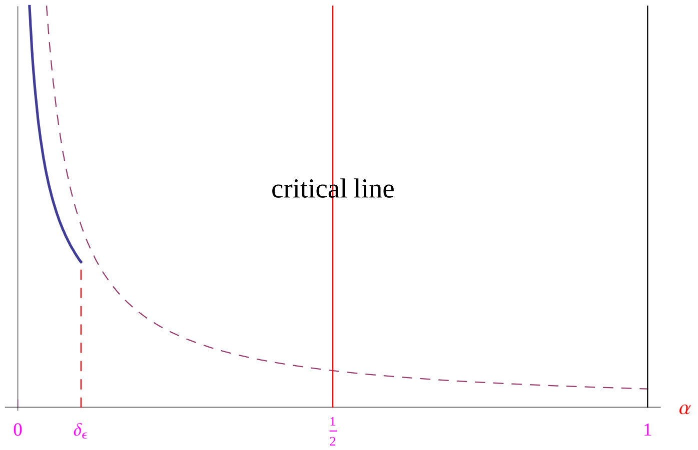


Figure 4: Andamento di $|I_\alpha(\omega)|$ in un intorno destro di $\alpha = 0$, confrontato con la $\frac{1}{2\alpha}$ (curva tratteggiata).

Definiamo

$$\varphi(\alpha) \stackrel{def}{=} \int_{-\infty}^{0} \frac{e^{\alpha t}}{e^{e^t}+1} dt \tag{22}$$

che è monotonamente decrescente in $(0, 1)$. Ciò si deduce dall'andamento della funzione integranda per diversi valori di $\alpha \in (0, 1)$ (fig. 5).

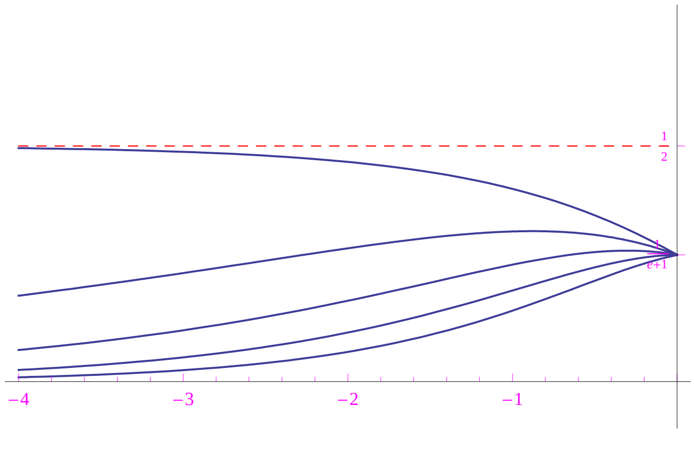


Figure 5: Andamento di $\frac{\varepsilon^{\alpha t}}{e^{e^t}+1}$ per $\alpha = 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$.

La (22) può essere interpretata come una rappresentazione integrale della funzione $\varphi(\alpha)$ che è elementarmente esprimibile solo per $\alpha = 1$. Infatti, in tal caso l'integrale si calcola facilmente:

$$\varphi(1) = \int_{-\infty}^{0} \frac{e^t}{e^{e^t}+1} dt = 1 + \ln 2 - \ln(1+e) \tag{23}$$

Ne consegue

$$\inf_{(0,1)} \varphi(\alpha) = 1 + \ln 2 - \ln(1+e), \quad \sup_{(0,1)} \varphi(\alpha) = +\infty$$

In fig. 6 l'andamento di $\varphi(\alpha)$.

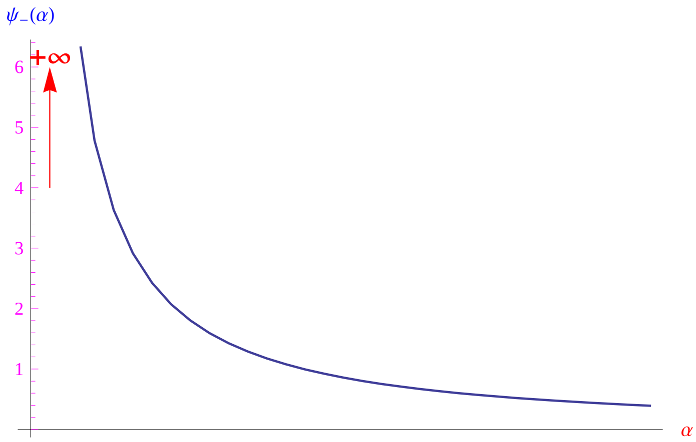


Figure 6: Andamento di $\varphi(\alpha)$ ottenuto per via numerica.

Passiamo a $\left|I_\alpha^{(+)}(\omega)\right|$.

$$\left|I_\alpha^{(+)}(\omega)\right| \leq \int_0^{+\infty} \frac{e^{\alpha t}}{e^{e^t}+1} dt \tag{24}$$

che suggerisce la posizione

$$\psi(\alpha) = \int_0^{+\infty} \frac{e^{\alpha t}}{e^{e^t}+1} dt \tag{25}$$

È facile convincersi che $\psi(\alpha)$ è monotonamente crescente in $(0,1)$. Quindi:

$$\begin{aligned} \inf_{(0,1)} \psi(\alpha) &= \psi(0) = \int_0^{+\infty} \frac{dt}{e^{e^t}+1} \simeq 0.180628 \\ \sup_{(0,1)} \psi(\alpha) &= \psi(1) = \int_0^{+\infty} \frac{e^t}{e^{e^t}+1} dt = \ln(1+e) - 1 \end{aligned} \tag{26}$$

Il grafico è in fig. 7, mentre in fig. 8 sono confrontati i grafici di singola funzione. (())

Da questa analisi segue immediatamente:

$$\alpha \in (0,1) \Longrightarrow \varphi(\alpha) > \psi(\alpha) \tag{27}$$

onde

$$\varphi(\alpha) > \ln(1+e) - 1, \quad \forall \alpha \in (0,1) \tag{28}$$

***

Ricerca degli zeri di $I_\alpha(\omega)$:

$$I_\alpha(\omega) = 0 \Longleftrightarrow I_\alpha^{(-)}(\omega) + I_\alpha^{(+)}(\omega) = 0 \Longleftrightarrow I_\alpha^{(-)}(\omega) = -I_\alpha^{(+)}(\omega) \Longrightarrow \left|I_\alpha^{(-)}(\omega)\right| = \left|I_\alpha^{(+)}(\omega)\right|$$

La (27) non fornisce informazione sull'esistenza degli zeri, giacché

$$\left|I_\alpha^{(-)}(\omega)\right| \leq \varphi(\alpha), \quad \left|I_\alpha^{(+)}(\omega)\right| \leq \psi(\alpha) \not\Rightarrow \exists (\alpha_0, \omega_0) \mid \left|I_{\alpha_0}^{(-)}(\omega_0)\right| = \left|I_{\alpha_0}^{(+)}(\omega_0)\right|$$

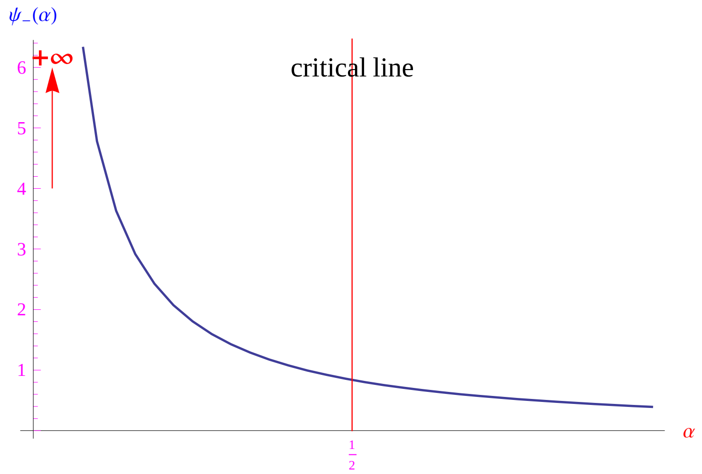


Figure 7: Andamento di $\psi(\alpha)$ ottenuto per via numerica.

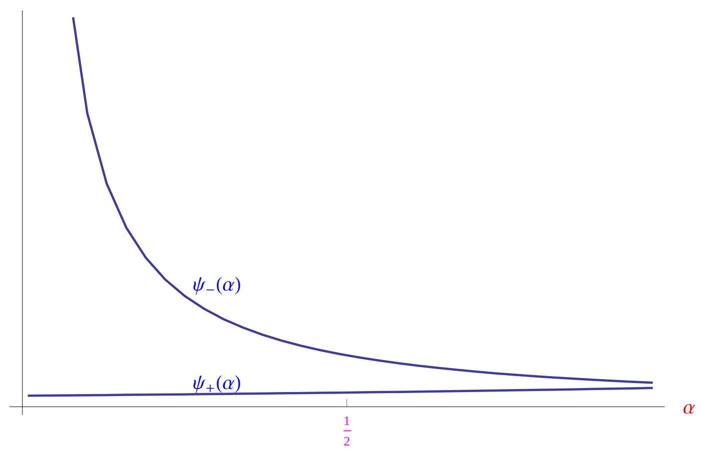


Figure 8: Andamento di $\varphi(\alpha), \psi(\alpha)$ ottenuto per via numerica.

References

[1] Titchmarsh E.C., Heath-Brown D.R., *The theory of the Riemann zeta-function* Clarendon Press - Oxford.

[2] Smirnov V.I. *Lezioni di Analisi Matematica, vol. II.* Editori Riuniti.