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$$\frac{d}{dx}f\left(x\right) \quad \sum_{k=0}^{+\infty}a_k \quad \int f\left(x\right)dx \quad \oint_{\Gamma}\left(Xdx+Ydy+Zdz\right)$$

The Riemann Zeta function and the Fourier Transform

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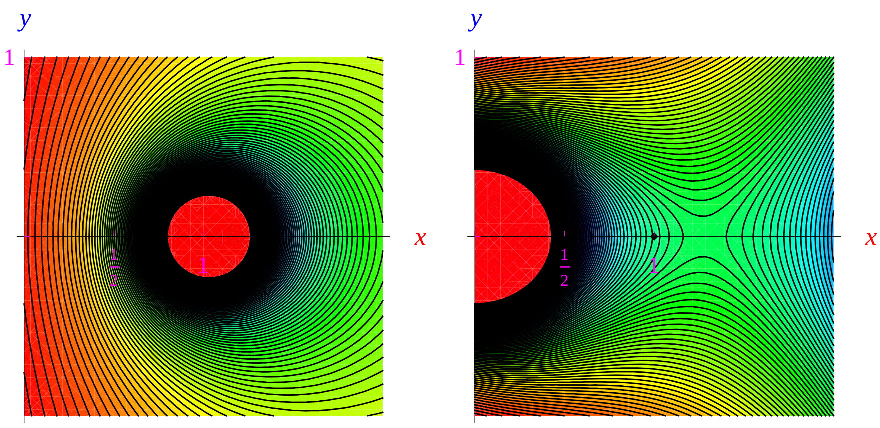

Contents

1 Introduction

The Dirichlet series $\sum_{n=1}^{+\infty} n^{-z}$ converges for $\operatorname{Re} z > 1$, and the sum is the *Riemann zeta function*:

$$\zeta(z) = \sum_{n=1}^{+\infty} \frac{1}{n^z}, \quad \operatorname{Re} z > 1 \tag{1}$$

Another notable series that can be expressed through the zeta function is:

$$\sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^z} = \left(1 - 2^{1-z}\right) \zeta(z)$$

which converges for $\operatorname{Re} z > 0$.

Since the series are not very "handy" it is preferable to work with integral representations.

2 A remarkable integral representation

In Quantum Statistical Mechanics the following generalized integrals which are not elementary expressible often appear

$$\int_0^{+\infty} \frac{t^{x-1} dt}{e^t \pm 1} \tag{2}$$

having:

$$\int_0^{+\infty} \frac{t^{x-1} dt}{e^t + 1} = \left(1 - 2^{1-x}\right) \Gamma(x) \zeta(x), \quad \forall x \in (0, +\infty) \tag{3}$$
$$\int_0^{+\infty} \frac{t^{x-1} dt}{e^t - 1} = \Gamma(x) \zeta(x), \quad \forall x \in (1, +\infty)$$

where $\Gamma(x)$ and $\zeta(x)$ are the Eulerian gamma function and the Riemann zeta function, respectively. Through an elementary change of variable, the first integral becomes

$$\int_{-\infty}^{+\infty} \frac{e^{xt}}{e^{e^t} + 1} dt \tag{4}$$

We define

$$f(x,t) = \frac{e^{xt}}{e^{e^t} + 1}, \quad \begin{cases} x \in (0,1) & \text{parameter} \\ t \in (-\infty, +\infty) & \text{independent variable} \end{cases} \tag{5}$$

Taking into account the first of (3):

$$\hat{f}(x) \overset{def}{=} \int_{-\infty}^{+\infty} f(x,t)\, dt = \left(1 - 2^{1-x}\right) \Gamma(x) \zeta(x), \quad \forall x \in (0, +\infty), \tag{6}$$

Proceeding by extension to the complex field, we can define the following function:

$$\hat{f}(z) \equiv \hat{f}(x+iy) = \int_{-\infty}^{+\infty} f(x,t)\, e^{iyt} dt = \left(1 - 2^{1-z}\right) \Gamma(z) \zeta(z), \quad \operatorname{Re}(z) > 0 \tag{7}$$

The *non-trivial zeros* of $\zeta(z)$ fall in the *critical strip* [1] of the complex plane defined by

$$A = \{z \in \mathbb{C} \mid 0 < \operatorname{Re} z < 1, \ -\infty < \operatorname{Im} z < +\infty\} \tag{8}$$

Proposition 1

$$\left|\left(1-2^{1-z}\right)\Gamma\left(z\right)\right|>0, \quad \forall z\in A \tag{9}$$

Proof. The inequality (9) derives from the fact that the gamma function has no zeros [2], while $1-2^{1-z}$ is manifestly zero-free in A. ■

From the proposition just proved it follows $f\left(z\right)$ and $\zeta\left(z\right)$ have the same (non-trivial) zeros.

3 Riemann Hypothesis

3.1 Fourier Transform

From (7) we see that for a given $x \in (0,1)$ the complex function $f(x+iy)$ is the Fourier transform of (5).

Conjecture 2 *(***Riemann Hypothesis***)*
The non-trivial zeros of the function

$$\hat{f}(x+iy) = \int_{-\infty}^{+\infty} f(x,t)\, e^{iyt} dt \tag{10}$$

have real part $x = 1/2$.

Let us first study the behavior of the function $f(x,t)$ (given by (5)) which for each value of the parameter $x \in (0,1)$ is defined in $(-\infty, +\infty)$.

Sign and intersections with the axes

It turns out $g(x,t) > 0$, $\forall t \in (-\infty, +\infty)$ for which the graph of f lies in the semi-plane of the positive ordinates. It does not intersect the abscissa axis, while it does intersect the ordinate axis at $\left(0, (e+1)^{-1}\right)$.

Behavior at extremes

After calculations:

$$\lim_{t \to +\infty} f(x,t) = 0^{+}, \quad \forall x \in (0,1)$$

The order of infinitesimal:

$$\lim_{t \to +\infty} t^{\alpha} f(x,t) = 0^{+}, \quad \forall \alpha > 0 \qquad \text{(infinitesimal of infinitely large order)} \tag{11}$$

$$\lim_{t \to -\infty} f(x,t) = \begin{cases} \frac{1}{2}^{-}, & \text{if } x = 0 \\ 0^{+}, & \text{if } x > 0 \end{cases}$$

Precisely:

$$\lim_{t \to -\infty} t^{\alpha} f(x > 0, t) = 0^{+}, \quad \forall \alpha > 0 \tag{12}$$

Conclusion: for $|t| \to +\infty$ the function $f(x > 0, t)$ is an infinitesimal of order infinitely large, provided that it is $x > 0$.

First derivative

$$f'(x,t) \equiv \frac{\partial}{\partial t} f(x,t) = \frac{e^{xt}\left[x\left(e^{e^t}+1\right) - e^{t+e^t}\right]}{\left(e^{e^t}+1\right)^2}$$

For $x = 0$

$$f'(0,t) = -\frac{e^{t+e^t}}{\left(e^{e^t}+1\right)^2} < 0, \quad \forall t \in (-\infty, +\infty)$$

so the function is strictly decreasing.

For $x > 0$

$$f'(x,t) = 0 \Longleftrightarrow x\left(e^{e^t}+1\right) - e^{t+e^t} = 0 \tag{13}$$

which is solved numerically. After calculations, the root of the (13) è

$$0 < x < 1 \Longrightarrow t_{*}(x) \in [\sim -6.32, 0.2]$$

Some values for assigned $x \in (0,1)$:

$$\begin{aligned} t_* \left(\frac{1}{5}\right) &\simeq -1.07 \\ t_* \left(\frac{1}{4}\right) &\simeq -0.88 \\ t_* \left(\frac{1}{2}\right) &\simeq -0.30 \\ t_* \left(\frac{2}{3}\right) &\simeq -0.07 \\ t_* \left(\frac{3}{4}\right) &\simeq 0.02 \end{aligned}$$

The sign is

$$\begin{aligned} -\infty < t < t_* (x) &\Longrightarrow f' (x,t) > 0 \\ t_* (x) < t < +\infty &\Longrightarrow f' (x,t) < 0 \end{aligned}$$

Hence the function is strictly increasing in $(-\infty, t_* (x))$ is strictly decreasing in $(t_* (x), +\infty)$. So $t_* (x)$ is a point of relative maximum for

Second derivative

$$f'' (x,t) = \frac{e^{xt} \left[e^{2\left(e^t+t\right)} - e^{e^t+2t} + x^2 \left(1+e^{e^t}\right)^2 - (2x+1) \left(e^{t+e^t} + e^{2e^t+t}\right) \right]}{\left(1+e^{e^t}\right)^3} \tag{14}$$

For $x = 0$

$$f'' (0,t) = \frac{e^{2\left(e^t+t\right)} - e^{e^t+2t} - \left(e^{t+e^t} + e^{2e^t+t}\right)}{\left(1+e^{e^t}\right)^3}$$

which has a zero in $t'_* (x=0) \simeq 0.43$. The sign is

$$\begin{aligned} -\infty < t < t'_* (x=0) &\Longrightarrow f'' (0,t) < 0 \\ t'_* (x=0) < t < +\infty &\Longrightarrow f'' (0,t) > 0 \end{aligned}$$

It follows that the graph of $f (0,t)$ is convex in $(-\infty, t'_* (x=0))$ and concave in $(t'_* (x=0), +\infty)$. So $(0.43, 0.18)$ is an inflection point with an oblique tangent. In fig. 1 we report the graph of $f (0,t)$.

For $x > 0$ we perform a qualitative analysis. The parameter x controls the slope of the graph of $f (t)$ in $(-\infty, 0)$ since

$$\frac{\partial}{\partial t} e^{xt} = x e^{xt}$$

For $t \in (0, +\infty)$ the slope is controlled by e^{e^t} in denominator. This implies that the effects of the parameter x are felt for $t \in (-\infty, 0)$, while in $(0, +\infty)$ the trend is practically independent of this parameter. Fig. 2 plots $f (x,t)$ for increasing values of the parameter x starting from $x = 0$.

We rewrite (6)

$$F (x) = \int_{-\infty}^{+\infty} f (x,t)\, dt \tag{15}$$

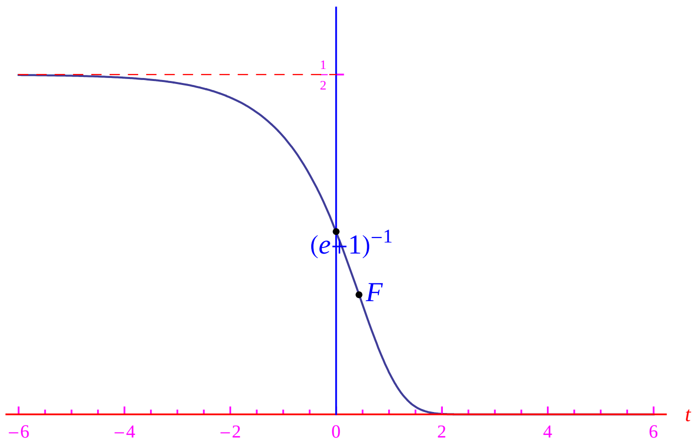

Figure 1: Trend of $f(0,t)$.

which *Mathematica* calculates through

$$F(x) = \left(1 - 2^{1-x}\right)\Gamma(x)\zeta(x)$$

As previously seen, for $x > 0$ the integrand function is for $t \to \pm\infty$ an infinitesimal of infinitely large order; so the integral converges. More precisely:

$$F(x) = \int_{-\infty}^{0} \frac{e^{xt}}{e^{e^t}+1}dt + \underbrace{\int_{0}^{+\infty} \frac{e^{xt}}{e^{e^t}+1}dt}_{\text{converges } \forall x \in (0,1)}$$

For $x = 0$

$$f(0,t) = \frac{1}{e^{e^t}+1} \underset{t\to-\infty}{\longrightarrow} \frac{1}{2} \Longrightarrow \int_{-\infty}^{0} \frac{dt}{e^{e^t}+1} = +\infty \Longrightarrow \lim_{x\to 0^+} F(x) = +\infty$$

For $x > 0$ the trend in $(-\infty, 0)$ is dominated by e^{xt}

$$\frac{e^{xt}}{e^{e^t}+1} \underset{t\to-\infty}{\longrightarrow} e^{xt}$$

so the integral converges. As x increases in $(-\infty, 0)$ the slope increases, and this favors the convergence of the integral[1], simultaneously decreases the area of the base trapezoid $(-\infty, 0)$ and therefore the value of $F(x)$. This shows that $G(x)$ is strictly decreasing, as confirmed by the graph fig. 3 obtained with *Mathematica*.

3.2 Zeros of the Fourier Transform

(5) can be interpreted as a quantity that depends deterministically on time t:

$$f(t) = e^{t/\tau}\nu(t) \tag{16}$$

where $\tau = x^{-1}$ plays the role of a time constant, while:

$$\nu(t) = \frac{1}{e^{e^t}+T}, \quad T = 1\,\mathrm{s} \tag{17}$$

[1]The parameter x therefore controls the speed of convergence of the integral in the interval $(-\infty, 0)$.

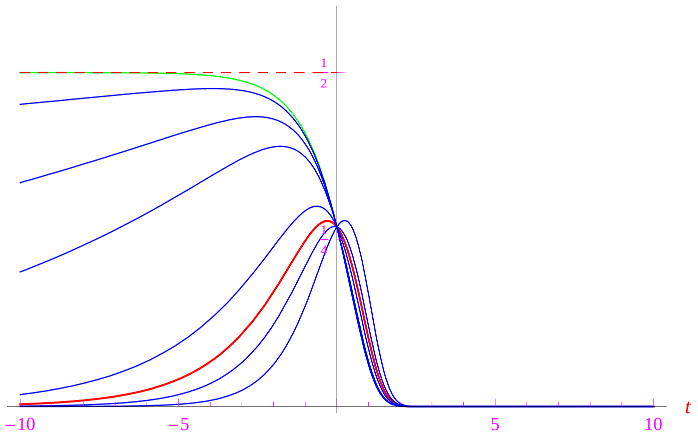

Figure 2: Trend of $f(x,t)$ for different values of x. Curve in green: $x=0$. The flattest curve towards the ordinate axis is for $x=1$.

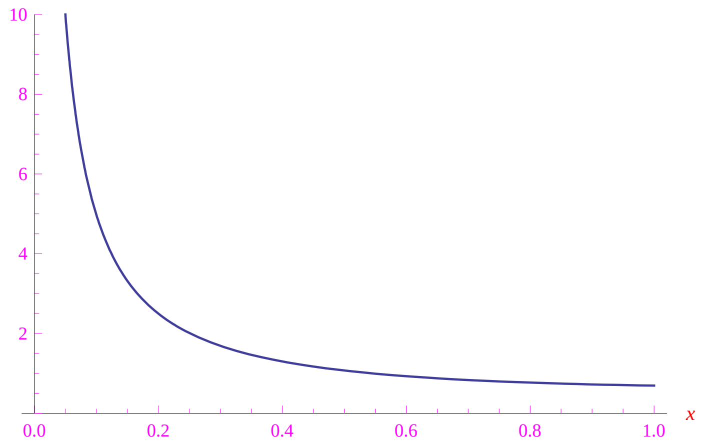

Figure 3: Trend of $F(x)$.

It follows that (16) has the dimensions of a frequency. Think – hypothetically – of a «signal» with a frequency that varies over time. Thus the (10) can be written as

$$\hat{f}_\tau\left(\Omega\right) = \int_{-\infty}^{+\infty} e^{t/\tau}\nu\left(t\right)e^{i\Omega t}dt \tag{18}$$

and is the Fourier transform of the «frequency» $f(t)$. By the Riemann Hypothesis:

$$\exists\Omega_0 \in R \mid \hat{f}_\tau\left(\Omega_0\right) = 0 \Longleftrightarrow \tau = 2 \tag{19}$$

References

[1] Titchmarsh E.C., Heath-Brown D.R., *The theory of the Riemann zeta-function* Clarendon Press - Oxford.

[2] Smirnov V.I. *Lezioni di Analisi Matematica, vol. II.* Editori Riuniti.