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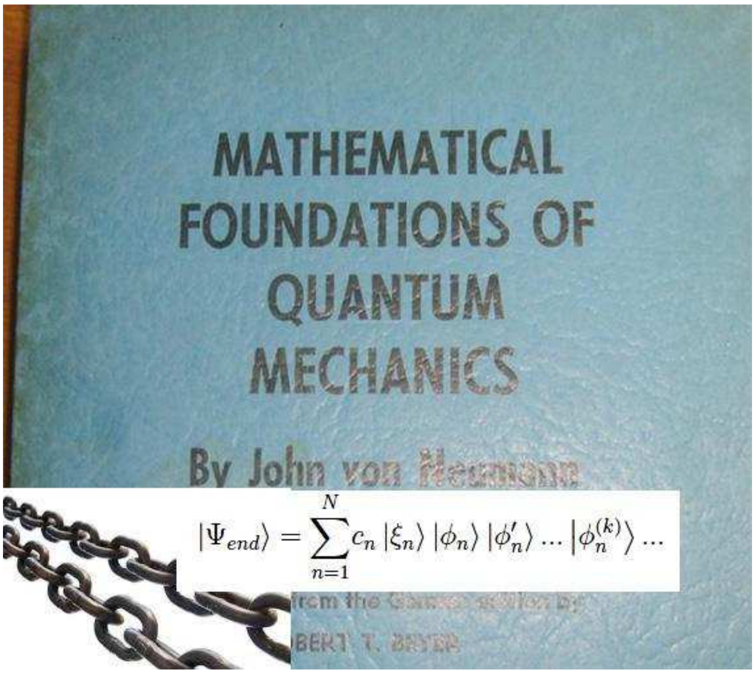
$$\frac{d}{dx}f(x) \quad \sum_{k=0}^{+\infty} a_k \quad \int f(x)\,dx \quad \oint_{\Gamma} (X\,dx + Y\,dy + Z\,dz)$$

The Wigner-Von Neumann Interpretation

Marcello Colozzo



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1 Dynamical evolution of a quantum system

Let S be a quantum system characterized by an observable X represented by a self-adjoint operator (in the appropriate Hilbert space $\mathcal{H}$) with a purely discrete spectrum, and with an eigenvalue equation[1]

$$\hat{X}\left|\xi_n\right\rangle = \xi_n\left|\xi_n\right\rangle, \quad n = 1, 2, ..., N \leq +\infty \tag{1}$$

Assuming the eigenvectors are normalized, we have:

$$\langle \xi_n | \xi_{n'} \rangle = \delta_{nn'},$$

so

$$\{|\xi_n\rangle\} = \text{orthonormal basis of } \mathcal{H}$$

Suppose the system is prepared at the initial time t_0, in a linear superposition of eigenstates of X:

$$\left|\psi_0\right\rangle = \sum_{n=1}^{N} c_n^{(0)}\left|\xi_n\right\rangle \tag{2}$$

$$c_n^{(0)} = \left\langle \psi_0 | \xi_n \right\rangle,$$

such that

$$\langle \psi_0 | \psi_0 \rangle = 1$$

or what is the same

$$\sum_{n=1}^{N}\left|c_n^{(0)}\right|^2 = 1$$

The time evolution at time t of the initial state is

$$\left|\psi\left(t\right)\right\rangle = \mathcal{U}\left(t, t_0\right)\left|\psi_0\right\rangle, \tag{3}$$

Here $\mathcal{U}\left(t, t_0\right)$ is a unitary operator, known as the *time evolution operator*. In the special case of an isolated system, i.e., with a Hamiltonian operator $\hat{H}$ that does not explicitly depend on time, we have:

$$\left|\psi\left(t\right)\right\rangle = e^{-\frac{i}{\hbar}(t-t_0)\hat{H}}\left|\psi_0\right\rangle \tag{4}$$

Let us recall that mathematically this is equivalent to solving the Schrödinger equation which in operator form is written:

$$\hat{H}\left|\psi\left(t\right)\right\rangle = i\hbar\frac{d}{dt}\left|\psi\left(t\right)\right\rangle \tag{5}$$

If the observable X is energy-compatible, that is, if the corresponding operators commute:

$$\left[\hat{X}, \hat{H}\right] = \hat{0},$$

is possible to explain the temporal evolution:

$$\left|\psi\left(t\right)\right\rangle = \sum_{n=1}^{N} c_n\left(t\right)\left|\xi_n\right\rangle \tag{6}$$

$$c_n\left(t\right) = c_n^{(0)} e^{-\frac{i}{\hbar}E_n(t-t_0)}$$

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[1] For simplicity and without loss of generality, let us assume that the eigenvalues are non-degenerate. ξ_n.

2 Measurement of a quantum observable

If at time $t_1 > t_0$, a measurement of the observable X is performed, the state of the system is *reduced* (or collapsed onto) one of the eigenstates of

$$|\psi(t_1)\rangle \underset{\text{measure of } X}{\longrightarrow} |\xi_k\rangle, \quad k \in \{1, 2, ..., N\},$$

with probability

$$P(t_1; \xi_k) = |c_k(t_1)|^2 \tag{7}$$

From what has just been said, it follows that the dynamical evolution of a quantum system occurs through two distinct processes:

1. **U-process**

   It is a unitary transformation of the Hilbert space itself. The unitary nature preserves the measurement probability of the individual eigenvalues. It is a continuous and linear (deterministic) process.

2. **R-process**

   It is the reduction of the state vector to one of the eigenstates of the observable being measured. It is a discontinuous and nondeterministic process.

3 Wigner-Von Neumann Interpretation

From an operational perspective, the measurement of a quantum observable occurs through interaction with a macroscopic apparatus (*measuring apparatus*). The definitive formulation of measurement theory was given by Von Neumann [1], according to which the measurement of an observable occurs through the implementation of a one-to-one correspondence between the properties of the quantum system S and those of a measuring apparatus M, whose macroscopicity is essential for it to be accessible to our senses.

Below is a summary of this approach, presented in [2].

The measuring apparatus M consists of an index that can be modeled by a classical particle P undergoing one-dimensional motion (x-axis). In general, the Hamiltonian of P takes the usual form:

$$H_P(x, p) = \frac{p^2}{2m} + V(x) \tag{8}$$

From a formal point of view, we can make the function $H_P(x, p)$ correspond to a self-adjoint operator in a suitable Hilbert space $\mathcal{H}_P$:

$$\hat{H}_P = H_P(\hat{x}, \hat{p}),$$

remembering that the impulse operator in the coordinate representation is given by

$$\hat{p} = -i\hbar\frac{d}{dx} \tag{9}$$

From this it follows that the Hamiltonian operator of the system $S + M$ is

$$\hat{H}_1(t) = \hat{H} + \hat{H}_P + \hat{H}_{int}(t), \tag{10}$$

where $\hat{H}$ is the Hamiltonian of S, while $\hat{H}_{int}(t)$ is the Hamiltonian describing the interaction between M and S. More precisely:

$$\hat{H}_{int}(t) = g(t)\,\hat{X}\hat{p},$$

being $g(t)$ a real function of time t. That being said, let us consider the special case in which the initial state of S is an eigenstate of X:

$$|\psi_0\rangle = |\xi_n\rangle\,, \qquad n \in \{1, 2, ..., N\} \tag{11}$$

The ket-vector:

$$|\phi_0\rangle \in \mathcal{H}_P$$

is the initial state of the particle P, so we move on to its initial wave function

$$\phi_0(x) = \langle x|\phi_0\rangle \in L^2(\mathbb{R}) \tag{12}$$

In order for P to model the index of a measuring device, it must be represented by a wave packet localized around a given point on the x-axis. To be more specific:

$$\phi_0(x) = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{+\infty}\hat{\phi}_0(p)\,e^{\frac{i}{\hbar}px}dp, \tag{13}$$

where

$$\hat{\phi}_0(p) = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{+\infty}\phi_0(x)\,e^{-\frac{i}{\hbar}px}dx,$$

that is, the wave function in the space of impulses (which is the Fourier transform of $\phi_0(x)$). As is known, the temporal evolution of the wave packet (13) determines a broadening of the same, which we can consider negligible with respect to the linear dimensions of the measuring apparatus. The initial state of the $S + M$ system is an element of the Hilbert space $\mathcal{H} \otimes \mathcal{H}_P$

$$|\Psi_0\rangle = |\xi_n\rangle\,|\phi_0\rangle\,,$$

whose time evolution at time $t \in [t_0, t_1]$, dove t_1 is the instant of measurement of the observable, is:

$$|\Psi(t)\rangle = e^{-\frac{i}{\hbar}(t-t_0)\left(\hat{H}+\hat{H}_P\right)}\,|\xi_n\rangle\,|\phi_0\rangle$$

Except for an inessential phase factor:

$$e^{-\frac{i}{\hbar}(t-t_0)\hat{H}}\,|\xi_n\rangle \to |\xi_n\rangle\,,$$

while

$$e^{-\frac{i}{\hbar}(t-t_0)\hat{H}_P}\,|\phi_0\rangle = |\phi(t)\rangle$$

From the above, the broadening of the wave packet is negligible, and so will be the temporal evolution for which we can assume a phase

$$e^{-\frac{i}{\hbar}(t-t_0)\hat{H}_P}\,|\phi_0\rangle \to |\phi_0\rangle$$

Finally:

$$|\Psi(t)\rangle = |\xi_n\rangle\,|\phi_0\rangle\,, \quad \forall t \in [t_0, t_1] \tag{14}$$

At time $t = t_1$ the interaction is “turned on” so the total Hamiltonian operator is given by (10). Due to the explicit time dependence exhibited by the Hamiltonian operator, it becomes difficult to explicitly express the time evolution operator:

$$|\Psi(t)\rangle = \mathcal{U}(t, t_1)\,|\xi_n\rangle\,|\phi_0\rangle\,, \quad \forall t \in [t_1, t_1']\,,$$

where $t_1' > t_1$ is the instant at which the measurement operation ends. We make the following reasonable assumption: the interaction is intense and short-lived. This last circumstance allows us to free ourselves from the explicit dependence on time, since we can replace the function $g(t)$ with its average value in the interval $\Delta t = t_1' - t_1$:

$$\bar{g} = \frac{1}{\Delta t}\int_{t_1}^{t_1'} g(t)\,dt$$

The high intensity of the interaction implies that $\hat{H}_{int}$ is the dominant term, so the Hamiltonian of the system can be written:

$$\hat{H}_1 \simeq \bar{g}\hat{X}\hat{p}, \tag{15}$$

so

$$|\Psi(t)\rangle = e^{-\frac{i}{\hbar}(t-t_0)\bar{g}\hat{X}\hat{p}}\,|\xi_n\rangle\,|\phi_0\rangle\,, \quad \forall t \in [t_1, t_1'] \tag{16}$$

But $|\xi_n\rangle$ is eigenket of $\hat{X}$

$$|\Psi(t)\rangle = e^{-\frac{i}{\hbar}(t-t_0)\bar{g}\xi_n\hat{p}}\,|\xi_n\rangle\,|\phi_0\rangle\,, \quad \forall t \in [t_1, t_1'] \tag{17}$$

At the end of the measurement

$$|\Psi(t_1')\rangle = e^{-\frac{i}{\hbar}\bar{g}'\xi_n\hat{p}}\,|\xi_n\rangle\,|\phi_0\rangle\,, \tag{18}$$

Here

$$\bar{g}' = \bar{g}\Delta t$$

Follows

$$|\Psi(t_1')\rangle = |\xi_n\rangle\left(\tau(\bar{g}'\xi_n)\,|\phi_0\rangle\right), \tag{19}$$

where

$$\tau(\bar{g}'\xi_n) = e^{-\frac{i}{\hbar}\bar{g}'\xi_n\hat{p}}$$

We recognize in this last equation the translation operator in the direction of p (hence of the x-axis, since the motion of the particle P is one-dimensional) and of amplitude $\bar{g}'\xi_n$. Therefore the translated ket is

$$\tau(\bar{g}'\xi_n)\,|\phi_0\rangle = |\phi_{\xi_n}\rangle\,,$$

while the wave function in the coordinate representation is written:

$$\langle x|\phi_{\xi_n}\rangle = \langle x - \bar{g}'\xi_n|\phi_0\rangle = \phi_0(x - \bar{g}'\xi_n)$$

so that following the measurement operation, the index has moved by an amount proportional to the eigenvalue ξ_n which is clearly the result of the measurement, since the system is initially prepared in $|\xi_n\rangle$. In this way, the (19) is rewritten

$$|\Psi(t_1')\rangle = |\xi_n\rangle\,|\phi_n\rangle\,, \tag{20}$$

where to lighten the notation, we have placed:

$$|\phi_n\rangle \equiv |\phi_{\xi_n}\rangle$$

At this point it is immediate to generalize (20) to the case in which the initial state is given by

$$|\psi_0\rangle = \sum_{n=1}^{N} c_n |\xi_n\rangle$$

Precisely, the state of the $S+M$ system at times $t < t_1$ is

$$|\Psi(t)\rangle = \sum_{n=1}^{N} c_n |\xi_n\rangle |\phi_0\rangle, \quad \forall t \in [t_0, t_1] \tag{21}$$

Using the previous arguments and taking into account the linearity of the time evolution operator (and therefore, of the Schrödinger equation), we have that at the end of the measurement the state ket of the composite system $S+M$ is

$$|\Psi(t_1')\rangle = \sum_{n=1}^{N} c_n \underbrace{e^{-\frac{i}{\hbar}(t-t_0)\bar{g}\xi_n\hat{p}} |\xi_n\rangle |\phi_0\rangle}_{|\xi_n\rangle|\phi_n\rangle}$$

i.e.

$$|\Psi(t_1')\rangle = \sum_{n=1}^{N} c_n |\xi_n\rangle |\phi_n\rangle \tag{22}$$

It follows that in the general case (initial state as a superposition of eigenstates) the (macroscopic) index observable is included in the linear superposition. To resolve this superposition, we include a second measuring apparatus M': if $t_2' > t_1'$ is the instant at which the second measurement ends, we have that the state ket of $S+M+M'$ is

$$|\Psi(t_2')\rangle = \sum_{n=1}^{N} c_n |\xi_n\rangle |\phi_n\rangle |\phi_n'\rangle \tag{23}$$

This procedure can be iterated, contemplating an infinite chain of measuring devices, so that at the end

$$|\Psi_{end}\rangle = \sum_{n=1}^{N} c_n |\xi_n\rangle |\phi_n\rangle |\phi_n'\rangle \ldots \left|\phi_n^{(k)}\right\rangle \ldots \tag{24}$$

It is therefore evident that this procedure does not resolve linear superposition. In other words, the linear combination of eigenstates of the observable (as the initial state) is invariant under measurement. The only way to render this procedure finite, and thus achieve an effective measurement, is to insert a measuring device capable of reading its own state. Such an apparatus is clearly endowed with introspective capacity. More precisely, according to Neumann, such a device includes the sense organs through which the observer observes the apparatus, the nervous system that sends sensory data to the brain, and so on, so that it is ultimately the observer's consciousness or faculty of introspection that allows him to know his own state, thus severing the chain.

$$M, M', \ldots M^{(k)}, \ldots,$$

which will symbolically take the form

$$M, M', ...M^{(k)}, ...\Omega$$

where Ω is the observer. By [3]:

*The subjective nature of this knowledge is evident; however, for the proponents of the Wigner-Von Neumann interpretation it does not constitute a difficulty since ≪is not (therefore) a mysterious interaction between the measuring apparatus and the object that produces the appearance of a new ψ in the system in the measurement. It is simply the consciousness of an "I" that separates itself from the primitive ψ function and constitutes a new objectivity by virtue of its conscious observation, thereby attributing a new wave function to the object.≫ (*F. London, E. Bauer*)*

References

[1] Von Neumann, J., *Mathematical Foundations of Quantum Mechanics.* Springer, 1932.

[2] Cohen-Tannoudji, C., *Quantum Mechanics.* Wiley-VCH

[3] Caldirola P. *Dalla microfisica alla macrofisica.* Biblioteca della EST, 1975