Esercizio di Analisi 1

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Esercizio 1 *Calcolare:*

$$\lambda = \lim_{x \to \frac{\pi}{2}} \frac{1 - \sin x}{\cos x \left(\cos \frac{x}{2} - \sin \frac{x}{2}\right)} \tag{1}$$

Soluzione

Il rapporto si presenta nella forma indeterminata $\frac{0}{0}$. Scriviamo:

$$\lambda = \lim_{x \to \frac{\pi}{2}} \frac{(1 - \sin x)\left(\cos \frac{x}{2} + \sin \frac{x}{2}\right)}{\cos x \left(\cos^2 \frac{x}{2} - \sin^2 \frac{x}{2}\right)} \tag{2}$$

Osservando che $\cos x = \cos^2 \frac{x}{2} - \sin^2 \frac{x}{2}$

$$\begin{aligned} \lambda &= \lim_{x \to \frac{\pi}{2}} \frac{(1 - \sin x)\left(\cos \frac{x}{2} + \sin \frac{x}{2}\right)}{\cos^2 x} \\ &= \lim_{x \to \frac{\pi}{2}} \frac{(1 - \sin x)\left(\cos \frac{x}{2} + \sin \frac{x}{2}\right)}{1 - \sin^2 x} \\ &= \lim_{x \to \frac{\pi}{2}} \frac{(1 - \sin x)\left(\cos \frac{x}{2} + \sin \frac{x}{2}\right)}{(1 - \sin x)(1 + \sin x)} \\ &= \lim_{x \to \frac{\pi}{2}} \frac{\cos \frac{x}{2} + \sin \frac{x}{2}}{1 + \sin x} \\ &= \frac{\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}}{1 + 1} = \frac{\sqrt{2}}{2} \end{aligned} \tag{3}$$

Alternativamente, poniamo $t = \frac{\pi}{2} - x$, per cui $\sin x = \cos t$, $\cos x = \sin t$, e applicando le formule di addizione:

$$\begin{aligned} \cos \frac{x}{2} &= \cos\left(\frac{\pi}{4} - \frac{t}{2}\right) = \cos \frac{\pi}{4} \cos \frac{t}{2} + \sin \frac{\pi}{4} \sin \frac{t}{2} = \frac{1}{\sqrt{2}}\left(\cos \frac{t}{2} + \sin \frac{t}{2}\right) \\ \sin \frac{x}{2} &= \sin\left(\frac{\pi}{4} - \frac{t}{2}\right) = \sin \frac{\pi}{4} \cos \frac{t}{2} - \cos \frac{\pi}{4} \sin \frac{t}{2} = \frac{1}{\sqrt{2}}\left(\cos \frac{t}{2} - \sin \frac{t}{2}\right) \\ &\Longrightarrow \cos \frac{x}{2} - \sin \frac{x}{2} = \frac{2}{\sqrt{2}} \sin \frac{t}{2} \end{aligned}$$

Segue

$$\lambda = \lim_{t \to 0} \frac{1 - \cos t}{\sin t \cdot \frac{2}{\sqrt{2}} \sin \frac{t}{2}}$$

Dividiamo numeratore e denominatore per t^2

$$\lambda = \frac{\sqrt{2}}{2} \lim_{t \to 0} \frac{\frac{1 - \cos t}{t^2}}{\frac{\sin t}{t} \cdot \frac{\sin \frac{t}{2}}{\frac{t}{2}} \cdot 2} = \sqrt{2} \frac{\frac{1}{2}}{1 \cdot 1} = \frac{\sqrt{2}}{2}$$