

Potenza ennesima di una matrice triangolare

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Esercizio 1 *Dimostrare*

$$\begin{aligned} A &= \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \implies A^p = \begin{pmatrix} 1 & pa \\ 0 & 1 \end{pmatrix} \\ B &= \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix} \implies B^q = \begin{pmatrix} 1 & 0 \\ qb & 1 \end{pmatrix} \end{aligned}, \quad p, q \in \mathbb{N}$$

Soluzione

Per definizione di potenza ennesima di una matrice quadrata

$$\begin{aligned} A^p &= \underbrace{\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \cdots \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}}_p \\ &= \begin{pmatrix} 1 & 2a \\ 0 & 1 \end{pmatrix} \underbrace{\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \cdots \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}}_{p-2} \\ &= \begin{pmatrix} 1 & 3a \\ 0 & 1 \end{pmatrix} \underbrace{\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \cdots \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}}_{p-3} \\ &= \begin{pmatrix} 1 & 4a \\ 0 & 1 \end{pmatrix} \underbrace{\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \cdots \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}}_{p-4} \\ &= \dots \\ &= \begin{pmatrix} 1 & pa \\ 0 & 1 \end{pmatrix} \end{aligned} \tag{1}$$

Per la matrice B procediamo in maniera simile:

$$\begin{aligned} B^q &= \underbrace{\begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix} \cdots \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}}_q \\ &= \begin{pmatrix} 1 & 0 \\ 2b & 1 \end{pmatrix} \underbrace{\begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix} \cdots \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}}_{q-2} \\ &= \begin{pmatrix} 1 & 0 \\ 3b & 1 \end{pmatrix} \underbrace{\begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix} \cdots \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}}_{q-3} \\ &= \begin{pmatrix} 1 & 0 \\ 4b & 1 \end{pmatrix} \underbrace{\begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix} \cdots \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}}_{q-4} \\ &= \dots \\ &= \begin{pmatrix} 1 & 0 \\ qb & 1 \end{pmatrix} \end{aligned} \tag{2}$$