Calcolo di una serie di Fourier

Marcello Colozzo – http://www.extrabyte.info

Sia data la funzione:

$$v(t) = \begin{cases} 0, & \text{se } 0 \le t \le \frac{\pi}{2\omega}, \ \frac{\pi}{\omega} \le t \le \frac{2\pi}{\omega} \\ V \sin \omega t, & \text{se } \frac{\pi}{2\omega} \le t \le \frac{\pi}{\omega} \end{cases}, \tag{1}$$

il cui grafico è riportato in fig. 1. Lo sviluppo in serie di Fourier è:

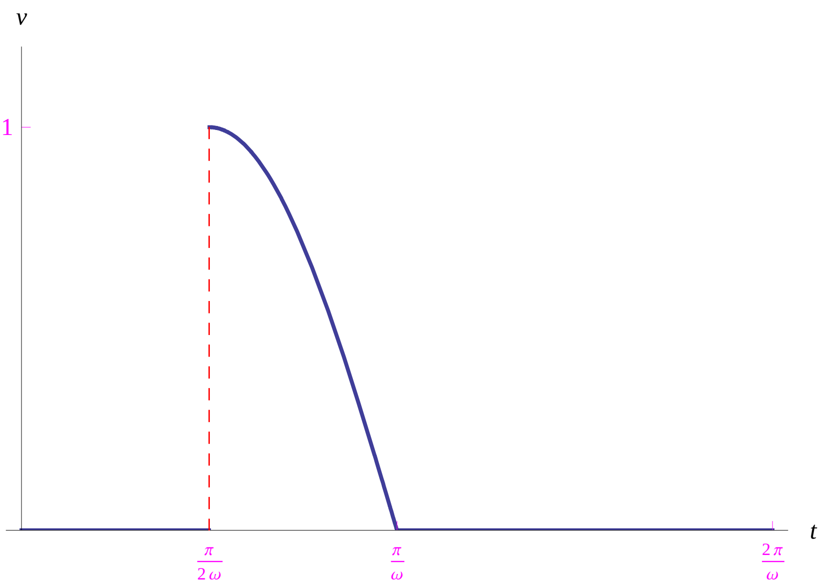

Figura 1: *Andamento della funzione (1)nell'intervallo di periodicità* $\left[0, \frac{2\pi}{\omega}\right]$.

$$v(t) = \frac{V_0}{2} + \sum_{k=1}^{+\infty} \left[V_k \cos\left(\frac{2k\pi t}{T}\right) + W_k \sin\left(\frac{2k\pi t}{T}\right) \right], \tag{2}$$

con

$$V_k = \frac{2}{T} \int_0^T v(t) \cos\left(\frac{2k\pi t}{T}\right) dt, \qquad (k = 0, 1, 2, ...)$$

$$W_k = \frac{2}{T} \int_0^T v(t) \sin\left(\frac{2k\pi t}{T}\right) dt, \qquad (k = 1, 2, ...)$$

Cioè

$$V_k = \frac{\omega V}{\pi} \int_{\frac{\pi}{2\omega}}^{\frac{\pi}{\omega}} \sin \omega t \cos(k\omega t)\, dt \tag{3}$$

$$W_k = \frac{\omega V}{\pi} \int_{\frac{\pi}{2\omega}}^{\frac{\pi}{\omega}} \cos \omega t \cos(k\omega t)\, dt$$

Eseguendo il cambio di variabile $x = \omega t$

$$\begin{aligned} V_k &= \frac{V}{\pi} I_k \\ W_k &= \frac{V}{\pi} J_k, \end{aligned} \tag{4}$$

dove:

$$\begin{aligned} I_k &= \int_{\frac{\pi}{2}}^{\pi} \sin x \cos kx dx, \qquad (k = 0, 1, 2, ...) \\ J_k &= \int_{\frac{\pi}{2}}^{\pi} \cos x \cos kx dx, \qquad (k = 1, 2, ...) \end{aligned} \tag{5}$$

Calcoliamo I_k. Per le formule di Werner:

$$\sin x \cos kx = \frac{1}{2} \left[\sin \left(1 + k \right) x + \sin \left(1 - k \right) x \right]$$

Quindi:

$$\begin{aligned} I_k &= \frac{1}{2} \int_{\frac{\pi}{2}}^{\pi} \left[\sin \left(1 + k \right) x + \sin \left(1 - k \right) x \right] dx \\ &= \frac{1}{2} \left[- \frac{1}{1+k} \cos \left(1 + k \right) x |_{x=\frac{\pi}{2}}^{x=\pi} - \frac{1}{1-k} \cos \left(1 - k \right) x |_{x=\frac{\pi}{2}}^{x=\pi} \right] \\ &= \frac{1}{2} \left[- \frac{1}{1+k} \left(\cos \left(1 + k \right) \pi - \cos \left(1 + k \right) \frac{\pi}{2} \right) - \frac{1}{1-k} \left(\cos \left(1 - k \right) \pi - \cos \left(1 - k \right) \frac{\pi}{2} \right) \right] \end{aligned} \tag{6}$$

Riesce:

$$\begin{aligned} \cos \left(1 + k \right) \pi &= \cos \left(\pi + k\pi \right) \\ &= - \cos k\pi = \\ &= - \left(-1 \right)^k \\ &= \left(-1 \right)^{k-1} \end{aligned}$$

Allo stesso modo:

$$\cos \left(1 - k \right) x = \left(-1 \right)^{k-1}$$

Inoltre:

$$\begin{aligned} \cos \left(1 + k \right) \frac{\pi}{2} &= \cos \left(\frac{\pi}{2} + k \frac{\pi}{2} \right) \\ &= - \sin \left(\frac{k\pi}{2} \right) \\ \cos \left(1 - k \right) \frac{\pi}{2} &= \cos \left(\frac{\pi}{2} - k \frac{\pi}{2} \right) \\ &= \sin \left(k \frac{\pi}{2} \right) \end{aligned}$$

Sostituendo tali risultati nella (6):

$$I_k = \frac{1}{1-k^2}\left[k\sin\left(k\frac{\pi}{2}\right) - (-1)^{k-1}\right] \tag{7}$$

In particolare:

$$I_0 = +1 \Longrightarrow V_0 = \frac{V}{\pi} \tag{8}$$

Si noti che la (7) per $k = 1$, per cui calcoliamo direttamente I_1:

$$\begin{aligned} I_1 &= \int_{\frac{\pi}{2}}^{\pi} \sin x \cos x dx = \frac{1}{2}\int_{\frac{\pi}{2}}^{\pi} \sin 2x dx \\ &= \frac{1}{4}\int_{\frac{\pi}{2}}^{\pi} \sin 2x d\left(2x\right) \\ &= -\frac{1}{2}, \end{aligned}$$

onde:

$$V_1 = -\frac{V}{2\pi} \tag{9}$$

Passiamo agli integrali J_k. Nuovamente Werner:

$$\sin x \sin kx = \frac{1}{2}\left[\cos\left(1-k\right)x - \cos\left(1+k\right)x\right],$$

perciò

$$\begin{aligned} J_k &= \frac{1}{2}\left[\int_{\frac{\pi}{2}}^{\pi} \cos\left(1-k\right)x dx - \int_{\frac{\pi}{2}}^{\pi} \cos\left(1+k\right)x dx\right] \qquad (10) \\ &= \frac{1}{2}\left[\frac{1}{1-k}\sin\left(1-k\right)x\Big|_{x=\frac{\pi}{2}}^{x=\pi} - \frac{1}{1+k}\sin\left(1+k\right)x\Big|_{x=\frac{\pi}{2}}^{x=\pi}\right] \\ &= \frac{1}{2}\left[\frac{1}{1-k}\left(\sin\left(1-k\right)\pi - \sin\left(1-k\right)\frac{\pi}{2}\right) - \frac{1}{1-k}\left(\sin\left(1+k\right)\pi - \sin\left(1+k\right)\frac{\pi}{2}\right)\right] \end{aligned}$$

Osserviamo che

$$\begin{aligned} \sin\left(1-k\right)\pi &= \sin\left(1+k\right)\pi = 0 \\ \sin\left(1-k\right)\frac{\pi}{2} &= \sin\left(\frac{\pi}{2} - \frac{k\pi}{2}\right) = \cos\left(k\frac{\pi}{2}\right) \\ \sin\left(1+k\right)\frac{\pi}{2} &= \sin\left(\frac{\pi}{2} + \frac{k\pi}{2}\right) = \cos\left(k\frac{\pi}{2}\right), \end{aligned}$$

onde

$$J_k = -\frac{k}{1-k^2}\cos\left(k\frac{\pi}{2}\right) \tag{11}$$

Anche qui dobbiamo calcolare direttamente J_1:

$$\begin{aligned} J_1 &= \int_{\frac{\pi}{2}}^{\pi} \sin^2 x dx = \frac{1}{2}\int_{\frac{\pi}{2}}^{\pi} (1-\cos 2x)\, dx \\ &= \frac{1}{2}\left(\int_{\frac{\pi}{2}}^{\pi} dx - \int_{\frac{\pi}{2}}^{\pi} \cos 2x dx\right) \\ &= \frac{1}{2}\left(\frac{\pi}{2}-0\right) \\ &= \frac{\pi}{4} \end{aligned}$$

Ciò implica

$$W_1 = \frac{V}{4}$$

Per ogni $k \in \{2, 3, ...\}$:

$$\begin{aligned} V_k &= \frac{V}{\pi\left(1-k^2\right)}\left[k\sin\left(k\frac{\pi}{2}\right) - (-1)^{k-1}\right] \\ W_k &= -\frac{Vk}{\pi\left(1-k^2\right)}\cos\left(k\frac{\pi}{2}\right) \end{aligned}$$

A questo punto riscriviamo lo sviluppo (2):

$$v(t) = \frac{V_0}{2} + V_1\cos\omega t + \sum_{k=2}^{+\infty} V_k \cos(k\omega t) + W_1 \sin(\omega t) + \sum_{k=2}^{+\infty} W_k \cos(k\omega t)$$

Cioè

$$\begin{aligned} v(t) = \frac{V}{2\pi} - \frac{V}{2\pi}\cos\omega t + \frac{V}{\pi}\sum_{k=2}^{+\infty}\frac{1}{1-k^2}\left[k\sin\left(k\frac{\pi}{2}\right) - (-1)^{k-1}\right]\cos(k\omega t) \\ + \frac{V}{4}\sin\omega t - \frac{V}{\pi}\sum_{k=2}^{+\infty}\frac{1}{1-k^2}\cos\left(k\frac{\pi}{2}\right)\sin(k\omega t) \end{aligned}$$

Al terzo ordine troviamo il grafico illustrato in fig. 2. Per $n = 20$ fig. 3.

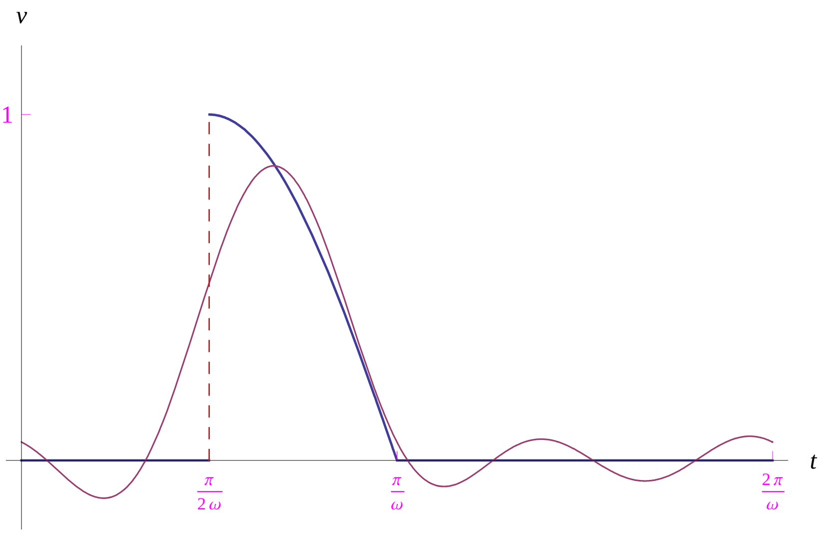

Figura 2: *Confronto di* $v(t)$ *con il suo sviluppo di Fourier troncato al terzo ordine.*

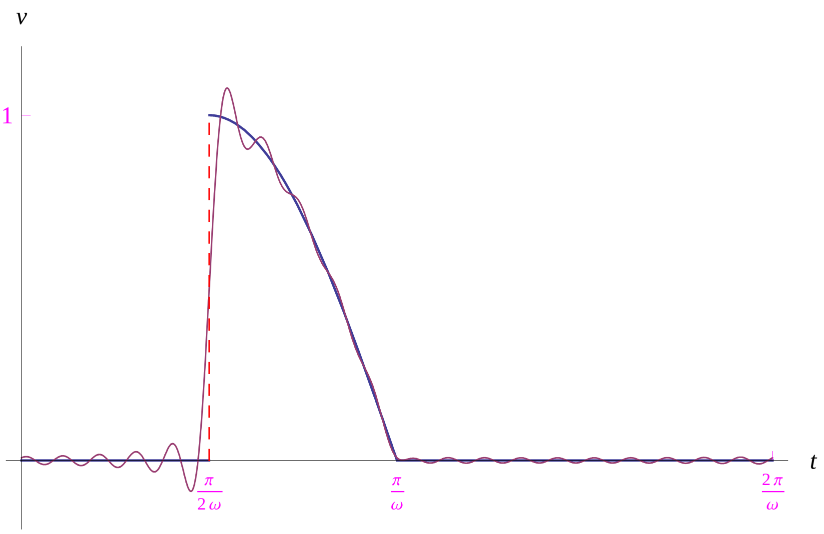

Figura 3: *Confronto di* $v(t)$ *con il suo sviluppo di Fourier troncato al ventesimo ordine.*