De Broglie hypothesis

Marcello Colozzo – http://www.extrabyte.info

As is known, electromagnetic radiation exhibits a dual nature: wave and particle. The first takes into account the phenomena of interference and diffraction. The second, however, intervenes in the processes of absorption (photoelectric effect) and diffusion (Compton effect). On the basis of these empirical truths, a plane and monochromatic electromagnetic wave of frequency ν and wavelength λ must be attributed an energy E and a momentum p given by the following relations

$$E = h\nu, \quad p = \frac{h}{\lambda}, \tag{1}$$

which can be rewritten in terms of Planck's reduced constant $\hbar = \frac{h}{2\pi}$:

$$E = \hbar\omega, \quad p = \hbar k, \tag{2}$$

being $\omega = 2\pi\nu$ and $k = \frac{2\pi}{\lambda}$ respectively the angular frequency and the number of waves. It follows that in all processes in which the corpuscular aspect occurs, the radiation behaves as a particle with energy $E = \hbar\omega$ and momentum $p = \hbar k$.According to De Broglie hypothesis, this correspondence is invertible: the motion of a particle with momentum p and energy E is equivalent to the propagation of a monochromatic plane wave. We recall that in the one-dimensional case the D'Alembert equation

$$\frac{\partial^2\psi}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = 0,$$

admits solutions of the plane wave type $\psi(x \pm ct)$. In the particular case of a monochromatic plane wave, in complex form we have:

$$\psi(x,t) = Ae^{i(kx-\omega t)}, \tag{3}$$

which generalizes to the three-dimensional case:

$$\psi(\mathbf{x},t) = Ae^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}, \tag{4}$$

where $\mathbf{k}$ is the vector defining the direction of propagation (*wave vector*), whose module is the number of waves $k = 2\pi/\lambda$ By the De Broglie hypothesis:

$$\mathbf{p} = \hbar\mathbf{k}, \quad \omega = \frac{E}{\hbar} \tag{5}$$

So

$$\psi(\mathbf{x},t) = Ae^{\frac{i}{\hbar}(\mathbf{p}\cdot\mathbf{x}-Et)}, \tag{6}$$

which propagates at the phase velocity:

$$v_f = \frac{\omega}{k} = \frac{E}{p} \tag{7}$$

If p is not exactly defined, the corresponding wave propagation coincides with the motion of a *wave packet*, i.e. a linear superposition of monochromatic plane waves, represented – by virtue of the linearity of D'Alembert equation – by the following expression :

$$\psi(\mathbf{x},t) = A\int_D e^{\frac{i}{\hbar}(\mathbf{p}\cdot\mathbf{x}-E(\mathbf{p})t)}d^3p, \tag{8}$$

where D is a bounded domain of the momentum space, where the linear momentum $\mathbf{p}$ of the particle is nonzero. From the theory of wave propagation, we know that the speed of the wave packet is the *group speed*:

$$v_g = \frac{d\omega}{dk} = \frac{d\omega}{dp}\frac{dp}{dk} = \frac{dE}{dp} \tag{9}$$

For example, for a non-relativistic particle:

$$E = \frac{p^2}{2m} + V(x) \Longrightarrow \frac{dE}{dp} = \frac{p}{m} = v, \tag{10}$$

for which

$$v_g = v \tag{11}$$

For the above, the function (6) is a solution of the D'Alembert equation.