Esercizio di Analisi 1

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Esercizio 1 *Dimostrare:*

$$\lim_{x\to x_0}\frac{\sin x-\sin x_0}{\sin\frac{x}{n}-\sin\frac{x_0}{n}}=\frac{n\cos x_0}{\cos\frac{x_0}{n}},\quad \forall n\in\mathbb{N}-\{0,1\}$$

Soluzione

Eseguendo il cambio di variabile $t=x-x_0$:

$$\sin x=\sin\left(x_0+t\right)=\sin x_0\cos t+\cos x_0\sin t$$
$$\sin\frac{x}{n}=\sin\left(\frac{x_0}{n}+\frac{t}{n}\right)=\sin\frac{x_0}{n}\cos\frac{t}{n}+\cos\frac{x_0}{n}\sin\frac{t}{n}$$

Il limite diventa

$$\begin{aligned}\lim_{t\to 0}\frac{\sin x_0\cos t+\cos x_0\sin t-\sin x_0}{\sin\frac{x_0}{n}\cos\frac{t}{n}+\cos\frac{x_0}{n}\sin\frac{t}{n}-\sin\frac{x_0}{n}}&=\lim_{t\to 0}\frac{\cos x_0\sin t-(1-\cos t)\sin x_0}{\cos\frac{x_0}{n}\sin\frac{t}{n}-\left(1-\cos\frac{t}{n}\right)\sin\frac{x_0}{n}}\\&=\lim_{t\to 0}\frac{\frac{\cos x_0\sin t-(1-\cos t)\sin x_0}{t}}{\frac{\cos\frac{x_0}{n}\sin\frac{t}{n}-\left(1-\cos\frac{t}{n}\right)\sin\frac{x_0}{n}}{t}}\\&=\lim_{t\to 0}\frac{\frac{\sin t}{t}\cos x_0-\frac{1-\cos t}{t}\sin x_0}{\frac{\sin\frac{t}{n}}{\frac{t}{n}}\cdot\frac{1}{n}\cos\frac{x_0}{n}-\frac{1-\cos\frac{t}{n}}{\frac{t}{n}}\cdot\frac{1}{n}\sin\frac{x_0}{n}}\\&=\frac{\cos x_0-0}{\frac{1}{n}\cos\frac{x_0}{n}},\end{aligned}$$

da cui l'asserto.