

Analytical mechanics

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Definition 1 Assigned a plane curve γ with a parametric representation:

$$x = x(t), \quad t \in (a, b), \quad \text{con } \mathbf{x}(t) \in C^{p \geq 1}$$

we define **length of** γ , the real positive number:

$$l = \int_a^b |\dot{\mathbf{x}}(t)| dt \quad (1)$$

This definition is consistent. For example, if γ is the circumference of center $O(0, 0)$ and radius R , its parametric representation is

$$\mathbf{x}(t) = R(\mathbf{i} \cos t + \mathbf{j} \sin t)$$

for which

$$|\dot{\mathbf{x}}(t)| = \sqrt{R^2 (\cos^2 t + \sin^2 t)} = R$$

Therefore

$$l = R \int_0^{2\pi} dt = 2\pi R$$

Theorem 2 The real number (1) is independent of the parametric representation.

Proof. Assigned $\gamma : \mathbf{x} = \mathbf{x}(t)$ we perform a *reparameterization* or a *substitution of an admissible parameter*:

$$t = t(\theta), \quad \text{con } t(\theta) \in C^2 \text{ e } \frac{dt}{d\theta} \neq 0 \quad (2)$$

Therefore

$$\gamma : \mathbf{x}(t(\theta)) = \mathbf{y}(\theta), \quad \theta \in (a', b'), \quad (3)$$

where

$$a' = \begin{cases} \theta(a), & \text{se } \frac{dt}{d\theta} > 0 \\ \theta(b), & \text{se } \frac{dt}{d\theta} < 0 \end{cases}, \quad b' = \begin{cases} \theta(b), & \text{se } \frac{dt}{d\theta} > 0 \\ \theta(a), & \text{se } \frac{dt}{d\theta} < 0 \end{cases} \quad (4)$$

Follows

$$l = \begin{cases} \int_{a'}^{b'} \left| \frac{d\mathbf{x}}{dt}(t(\theta)) \right| \frac{dt}{d\theta} d\theta, & \text{se } \frac{dt}{d\theta} > 0 \\ - \int_{b'}^{a'} \left| \frac{d\mathbf{x}}{dt}(t(\theta)) \right| \frac{dt}{d\theta} d\theta, & \text{se } \frac{dt}{d\theta} < 0 \end{cases} \implies l = \int_{a'}^{b'} \left| \frac{d\mathbf{x}}{dt}(t(\theta)) \right| \left| \frac{dt}{d\theta} \right| d\theta$$

But

$$\left| \frac{d\mathbf{x}}{dt}(t(\theta)) \right| \left| \frac{dt}{d\theta} \right| = \left| \frac{d\mathbf{y}}{d\theta} \right|$$

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References

- [1] Fasano A., Marmi S. 1994. *Meccanica analitica*. Boringhieri