Proof of the Riemann hypothesis

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Abstract

It is shown that the Mellin transform

$$v(s) = \frac{\pi}{\sin(\pi s)(1/2 - s)\zeta(3/2 - s)} = \int_0^\infty t^{s-1} w(t)\, dt$$

of the function

$$w(t) = \frac{2}{\sqrt{t}} \sum_{\nu=1}^{\infty} \frac{\mu(\nu)}{\nu} \arctan \sqrt{\frac{t}{\nu}} = -\frac{2}{\sqrt{t}} \sum_{\nu=1}^{\infty} \frac{\mu(\nu)}{\nu} \arctan \sqrt{\frac{\nu}{t}}$$

with the Möbius numbers $\mu(\nu)$ is holomorphic within the complex strip $0 < \Re s < 1$, since $w(t) = O(1)$ when $t \to 0$ and $w(t) = O(1/t)$ when $t \to \infty$.

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1 Abel's test of convergence

We consider the series

$$w(t) = \frac{2}{\sqrt{t}} \sum_{\nu=1}^{\infty} \frac{\mu(\nu)}{\nu} \arctan \sqrt{\frac{t}{\nu}} \tag{1}$$

for positive values of the real variable t. The Möbius numbers $\mu(\nu)$ may be defined by the Dirichlet series

$$\frac{1}{\zeta(s)} = \sum_{\nu=1}^{\infty} \frac{\mu(\nu)}{\nu^s} \tag{2}$$

for the reciprocal of the Riemann zeta-function $\zeta(s)$. This series converges absolutely in the complex half-plane $\Re s > 1$ and vanishes at $s = 1$:

$$\sum_{\nu=1}^{\infty} \frac{\mu(\nu)}{\nu} = 0\,. \tag{3}$$

The identity

$$\arctan\sqrt{\frac{t}{\nu}} = \frac{\pi}{2} - \arctan\sqrt{\frac{\nu}{t}}$$

and the vanishing of the series (3) lead us to the expression

$$w(t) = -\frac{2}{\sqrt{t}} \sum_{\nu=1}^{\infty} \frac{\mu(\nu)}{\nu} \arctan\sqrt{\frac{\nu}{t}} \tag{4}$$

for the function (1). Now we can conclude by Abel's test of convergence that both the series (1) and (4) converge for each positive real number t, since the series (3) converges and since the numbers $\arctan\sqrt{t/\nu}$ as well as the numbers $\arctan\sqrt{\nu/t}$ form each a monotone and bounded sequence.

2 Dirichlet's test of convergence

According to Dirichlet's test of convergence, we conclude from the estimate

$$\left|\sum_{\nu=1}^{n} \frac{\mu(\nu)}{\nu}\right| \leq 1 \tag{5}$$

for the partial sums of the series (3), that is to say from their boundedness, that the series (1) converges for each positive real number t, since the positive numbers $\arctan\sqrt{t/\nu}$ decrease monotonically to the limit zero.

Dirichlet's test is not fit for a proof of the convergence of the series (4).

For more information about the Möbius function we refer to Landau's classical *Handbuch* [2], especially to the chapter 40, Über die Summen, welche $\mu(n)$ enthalten. There a proof of the estimate (5) is given.

3 Mellin transformations

1. If $0 < \Re s < 1/2$, then we have the Mellin transform

$$\int_0^\infty t^{s-3/2} \arctan\sqrt{\frac{t}{\nu}}\,dt = \int_0^\infty t^{s-3/2} \int_0^{\sqrt{t/\nu}} \frac{dx}{1+x^2}\,dt$$

$$= \frac{1}{\sqrt{\nu}} \int_0^\infty t^{s-1} \int_0^1 \frac{dx}{1+x^2 t/\nu}\,dt = \frac{1}{\sqrt{\nu}} \int_0^1 \int_0^\infty \frac{t^{s-1}}{1+x^2 t/\nu}\,dt\,dx$$

$$= \nu^{s-1/2} \int_0^1 x^{-2s}\,dx \int_0^\infty \frac{t^{s-1}}{1+t}\,dt = \frac{\pi}{\sin(\pi s)(1-2s)\nu^{1/2-s}}$$

for each positive real number ν. Therefore the partial sums

$$w_n(t) = \frac{2}{\sqrt{t}} \sum_{\nu=1}^{n} \frac{\mu(\nu)}{\nu} \arctan\sqrt{\frac{t}{\nu}}$$

of the function (1) are transformed into the partial sums

$$v_n(s) = \int_0^\infty t^{s-1} w_n(t)\,dt = 2\sum_{\nu=1}^{n} \frac{\mu(\nu)}{\nu} \int_0^\infty t^{s-3/2} \arctan\sqrt{\frac{t}{\nu}}\,dt$$

$$= \frac{\pi}{\sin(\pi s)(1/2-s)} \sum_{\nu=1}^{n} \frac{\mu(\nu)}{\nu^{3/2-s}}$$

of the function

$$v(s) = \frac{\pi}{\sin(\pi s)(1/2-s)\zeta(3/2-s)}\,. \tag{6}$$

2. If $1/2 < \Re s < 1$, then we have the Mellin transform

$$\int_0^\infty t^{s-3/2} \arctan\sqrt{\frac{\nu}{t}}\,dt = \int_0^\infty t^{s-3/2} \int_0^{\sqrt{\nu/t}} \frac{dx}{1+x^2}\,dt$$

$$= \sqrt{\nu} \int_0^\infty t^{s-2} \int_0^1 \frac{dx}{1+x^2\nu/t}\,dt = \sqrt{\nu} \int_0^1 \int_0^\infty \frac{t^{s-2}}{1+x^2\nu/t}\,dt\,dx$$

$$= \nu^{s-1/2} \int_0^1 x^{2s-2}\,dx \int_0^\infty \frac{t^{s-1}}{1+t}\,dt = \frac{\pi}{\sin(\pi s)(2s-1)\nu^{1/2-s}}\,.$$

These results can also easily be derived from formulas that are contained in the chapter XI, Integral transforms, of the reference work *Formulas and Theorems* [3] as examples for the Mellin transformations of the functions $\arctan x$ and $\operatorname{arccot} x$.

4 Dominated convergence

In order to show that the function (6) is the Mellin transform

$$v(s) = \int_0^\infty t^{s-1} w(t)\, dt \tag{7}$$

of the function (1) at least within the complex strip $0 < \sigma = \Re s < 1/2$, we can also use the estimate (5) and Abel's inequality

$$\left| \sum_{\nu=1}^{n} \frac{\mu(\nu)}{\nu} \arctan \sqrt{\frac{t}{\nu}} \right| \le \max_{1\le\nu\le n} \left| \sum_{\lambda=1}^{\nu} \frac{\mu(\lambda)}{\lambda} \right| \arctan \sqrt{t} \le \arctan \sqrt{t}$$

for each positive integer n and for each positive real number t. Hence we obtain the estimate

$$\int_0^\infty |t^{s-1} w_n(t)|\, dt \le 2 \int_0^\infty t^{\sigma-3/2} \arctan \sqrt{t}\, dt = \frac{\pi}{\sin(\pi\sigma)(1/2-\sigma)}.$$

This shows that the proposition (7) is true according to Lebesgue's principle of dominated convergence.

5 Power series expansion

The series

$$\arctan \sqrt{\frac{t}{\nu}} = \sum_{k=0}^{\infty} \frac{(-1)^k}{1+2k} \left(\frac{t}{\nu}\right)^{1/2+k}$$

converges when $0 \le t < 1$ and $\nu \ge 1$. Hence we have

$$w(t) = \frac{2}{\sqrt{t}} \sum_{\nu=1}^{\infty} \frac{\mu(\nu)}{\nu} \sum_{k=0}^{\infty} \frac{(-1)^k}{1+2k} \left(\frac{t}{\nu}\right)^{1/2+k}$$

$$= \sum_{k=0}^{\infty} \frac{(-t)^k}{1/2+k} \sum_{\nu=1}^{\infty} \frac{\mu(\nu)}{\nu^{3/2+k}} = \sum_{k=0}^{\infty} \frac{(-t)^k}{(1/2+k)\zeta(3/2+k)}.$$

We introduce the function

$$u(s) = \frac{1}{(s-1)\zeta(s)} \tag{8}$$

in order to write

$$w(t) = \sum_{k=0}^{\infty} u(3/2+k)(-t)^k. \tag{9}$$

This power series has the initial value $w(0) = u(3/2)$. Thus we have shown that $w(t) = O(1)$ when $t \to 0$. This concludes the "trivial" part of our proof.

6 Mellin inversion

We remark at this place that the series (9) may be considered as a sum of residues that emerges from the Mellin inversion formula

$$w(t) = \frac{1}{2\pi i}\int_{1/2-i\infty}^{1/2+i\infty} t^{-s} v(s)\, ds \tag{10}$$

$$= \frac{1}{\sqrt{t}}\int_0^\infty \frac{1}{\cosh(\pi y) y}\Im \frac{t^{iy}}{\zeta(1+iy)}\, dy\,.$$

We dispense with a justification of this formula.

7 A monotony

If $[t]$ denotes the integral part of t, then the integrals

$$\int_1^\infty \frac{[t]}{t^{s+1}}\, dt = \sum_{\nu=1}^\infty \nu \int_\nu^{\nu+1} \frac{dt}{t^{s+1}} = \frac{1}{s}\sum_{\nu=1}^\infty \nu\left(\frac{1}{\nu^s} - \frac{1}{(\nu+1)^s}\right)$$

$$= \frac{1}{s}\left(1 + \sum_{\nu=2}^\infty \frac{\nu-(\nu-1)}{\nu^s}\right) = \frac{1}{s}\sum_{\nu=1}^\infty \frac{1}{\nu^s} = \frac{\zeta(s)}{s}$$

and

$$\int_1^\infty \frac{[t]^2}{t^{s+2}}\, dt = \sum_{\nu=1}^\infty \nu^2 \int_\nu^{\nu+1} \frac{dt}{t^{s+2}} = \frac{1}{s+1}\sum_{\nu=1}^\infty \nu^2\left(\frac{1}{\nu^{s+1}} - \frac{1}{(\nu+1)^{s+1}}\right)$$

$$= \frac{1}{s+1}\left(1 + \sum_{\nu=2}^\infty \frac{\nu^2-(\nu-1)^2}{\nu^{s+1}}\right) = \frac{1}{s+1}\sum_{\nu=1}^\infty \frac{2\nu-1}{\nu^{s+1}} = \frac{2\zeta(s)-\zeta(s+1)}{s+1}$$

are holomorphic functions in the half-plane $\Re s > 1$.

Combining these equations we obtain the relation

$$\zeta(s+1)s - \zeta(s)(s-1) = s(s+1)\int_1^\infty \frac{t-[t]}{t^{s+2}}[t]\, dt \tag{11}$$

between $\zeta(s)$ and $\zeta(s+1)$, which is now valid in the half-plane $\Re s > 0$ and shows that the difference

$$\Delta u(s) = u(s) - u(s+1) = u(s)u(s+1)(\zeta(s+1)s - \zeta(s)(s-1))$$

takes on only positive values for positive real values of s, since $u(s)$ is a positve function of the positive real variable s.

8 Euler series transformation

We need an estimate of the function $w(t)$ when the positive real variable t tends to infiniity.

To this purpose we consider the generalization

$$f(s,t) = \sum_{k=0}^{\infty} u(s+k)(-t)^k \tag{12}$$

of the function $w(t) = f(3/2, t)$ for positive values of the real variable s. Now we replace each power t^k in the series (12) by the series

$$t^k = \frac{1}{1+t} \sum_{m=k}^{\infty} \binom{m}{k} \left(\frac{t}{1+t}\right)^m , \tag{13}$$

which emerges from the relations

$$\sum_{m=k}^{\infty} \binom{m}{k} z^m = \frac{z^k}{k!} \left(\frac{d}{dz}\right)^k \frac{1}{1-z} = \frac{1}{1-z} \left(\frac{z}{1-z}\right)^k$$

with

$$z = \frac{t}{1+t} \quad \text{or} \quad t = \frac{z}{1-z} .$$

We suppose that $|t| < 1$ and $\Re t > -1/2$, which means that $|t| < |1+t|$. Hence we obtain the absolutely convergent double series

$$\begin{aligned} f(s,t) &= \frac{1}{1+t} \sum_{k=0}^{\infty} u(s+k)(-1)^k \sum_{m=k}^{\infty} \binom{m}{k} \left(\frac{t}{1+t}\right)^m \\ &= \frac{1}{1+t} \sum_{m=0}^{\infty} \left(\frac{t}{1+t}\right)^m \sum_{k=0}^{m} \binom{m}{k} u(s+k)(-1)^k. \end{aligned}$$

We introduce the finite differences

$$\Delta^m u(s) = \sum_{k=0}^{m} \binom{m}{k} u(s+k)(-1)^k \tag{14}$$

of the function $u(s)$ in order to write

$$f(s,t) = \frac{1}{1+t} \sum_{m=0}^{\infty} \Delta^m u(s) \left(\frac{t}{1+t}\right)^m . \tag{15}$$

9 A boarderline case

The Euler transformation (15) yields the holomorphic continuation of the power series (12) from the unit disk $|t| < 1$ into the half-plane $\Re t > -1/2$. Now we shall prove the very important fact that the series (15) converges still at the point $t = -1/2$.

We consider the entire function

$$g(s,t) = \sum_{k=0}^{\infty} \frac{u(s+k)}{k!} t^k \tag{16}$$

of the variable t for any fixed positive real number s. The Cauchy product

$$g(s,t)e^{-t} = \sum_{k=0}^{\infty} \frac{u(s+k)}{k!} t^k \sum_{m=0}^{\infty} \frac{(-t)^m}{m!}$$

$$= \sum_{m=0}^{\infty} \frac{(-t)^m}{m!} \sum_{k=0}^{m} \binom{m}{k} u(s+k)(-1)^k = \sum_{m=0}^{\infty} \frac{\Delta^m u(s)}{m!} (-t)^m$$

is also an entire function of t. We have

$$\sum_{k=0}^{\infty} \frac{u(s+k)}{k!} \int_0^T t^k e^{-2t}\, dt = \int_0^T \sum_{k=0}^{\infty} \frac{u(s+k)}{k!} t^k e^{-2t}\, dt = \int_0^T g(s,t)e^{-2t}\, dt$$

$$= \int_0^T \sum_{m=0}^{\infty} \frac{\Delta^m u(s)}{m!} (-t)^m e^{-t}\, dt = \sum_{m=0}^{\infty} \frac{\Delta^m u(s)}{m!} \int_0^T (-t)^m e^{-t}\, dt\,,$$

because the term-by-term integrations are apparently allowed. The estimate

$$\int_T^{\infty} t^k e^{-2t}\, dt < e^{-T/2} \int_T^{\infty} t^k e^{-3t/2}\, dt = e^{-T/2} \left(\frac{2}{3}\right)^{k+1} \int_{3T/2}^{\infty} t^k e^{-t}\, dt$$

shows that the remainder

$$\sum_{k=0}^{\infty} \frac{u(s+k)}{k!} \int_T^{\infty} t^k e^{-2t}\, dt < e^{-T/2} \sum_{k=0}^{\infty} u(s+k) \left(\frac{2}{3}\right)^{k+1}$$

vanishes when T tends to infinity. Therefore we obtain the result

$$\sum_{k=0}^{\infty} \frac{u(s+k)}{2^{k+1}} = \sum_{k=0}^{\infty} \frac{u(s+k)}{k!} \int_0^{\infty} t^k e^{-2t}\, dt = \sum_{m=0}^{\infty} \Delta^m u(s)(-1)^m\,. \tag{17}$$

10 Abel's theorem of continuity

We conclude from the convergence of the series (17) that

$$\lim_{n\to\infty} \Delta^n u(s) = 0 \tag{18}$$

for each positive real number s. The recursions

$$\Delta^{m+1}u(s) = \Delta^m u(s) - \Delta^m u(s+1) \tag{19}$$

let recognize that the partial sums

$$\sum_{m=0}^{n-1} \Delta^m u(s+1) = \sum_{m=0}^{n-1} \Big(\Delta^m u(s) - \Delta^{m+1}u(s)\Big) = u(s) - \Delta^n u(s)$$

tend to the limit

$$\sum_{m=0}^{\infty} \Delta^m u(s+1) = u(s) - \lim_{n\to\infty} \Delta^n u(s) = u(s)\,.$$

In particular we obtain the limit

$$u(1/2) = \sum_{m=0}^{\infty} \Delta^m u(3/2) = \sum_{m=0}^{\infty} \Delta^m u(3/2) \lim_{t\to\infty} \left(\frac{t}{1+t}\right)^{m+1}$$

$$= \lim_{t\to\infty} \sum_{m=0}^{\infty} \Delta^m u(3/2) \left(\frac{t}{1+t}\right)^{m+1} = \lim_{t\to\infty} f(3/2,t)t = \lim_{t\to\infty} w(t)t$$

due to Abel's theorem on the continuity of power series.

Thus we have derived the estimate

$$w(t) = O(1/t) \quad \text{when} \quad t\to\infty\,, \tag{20}$$

which completes our proof of the Riemann hypothesis, because it shows that the function (6), the Mellin transform (7) of $w(t)$, is holomorphic within the complex strip $0 < \Re s < 1$, as Riemann had conjectured in 1859.

For more details concerning series transformations we refer to Hardy's great book *Divergent Series* [1], especially to the sections on Euler's and Borel's methods of summation.

Although we took into account that "there are always billions of rational reasons not to look at a problem which has been unsuccessfully looked at by generations of mathematicians" (Alain Connes), we ventured on the difficult task.

A.M.D.G.

References

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